9.4 Intersection of three Planes

A Intersection of three Planes
Let consider three planes given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$
$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$
$$\pi_3 : A_3x + B_3y + C_3z + D_3 = 0$$

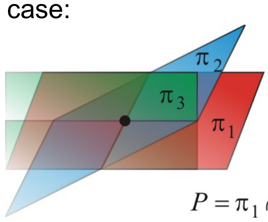
The point(s) of *intersection* of these planes is (are) related to the *solution(s)* of the following system of equations:

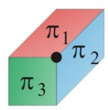
$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \quad (*)$$

There are *three* equations and *three* unknowns.
You may use *substitution* or *elimination* to solve this system.

B Unique Solution
(Point Intersection – Non Coplanar Normal Vectors)

In this case:





$P = \pi_1 \cap \pi_2 \cap \pi_3$

- ⇨ The planes *intersect* into a *single point*.
- ⇨ The *normal vectors* are *not coplanar*: $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$.
- ⇨ By solving the system (*), you get a *unique solution* for x , y , and z .

Ex 1. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x - 3y - 2z = -9 & (1) \\ 2x - 5y + z = 3 & (2) \\ -3x + 6y + 2z = 8 & (3) \end{cases}$$

Let isolate z from (2) and substitute in (1) and (3):

$$\begin{cases} x - 3y - 2(-2x + 5y + 3) = -9 & (4) \\ z = -2x + 5y + 3 & (5) \\ -3x + 6y + 2(-2x + 5y + 3) = 8 & (6) \end{cases}$$

$$\begin{cases} 5x - 13y = -3 & (4)\times 7 \\ z = -2x + 5y + 3 & (5) \\ -7x + 16y = 2 & (6)\times 5 \end{cases} \Rightarrow \begin{cases} 35x - 91y = -21 & (7) \\ z = -2x + 5y + 3 & (5) \\ -35x + 80y = 10 & (8) \end{cases}$$

Let add (7) and (8) to eliminate x :

$-11y = -11 \quad \Rightarrow y = 1 \quad (9)$

Let substitute (9) into (4) to get x :

$5x - 13(1) = -3 \Rightarrow 5x = 10 \Rightarrow x = 2 \quad (10)$

Let substitute (9) and (10) into (5) to get z :

$z = -2(2) + 5(1) + 3 = -4 + 5 + 3 = 4 \quad (11)$

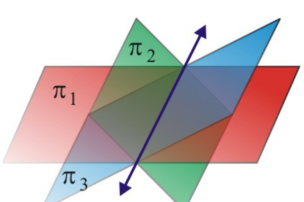
$\pi_1 \cap \pi_2 \cap \pi_3 = P(2,1,4)$

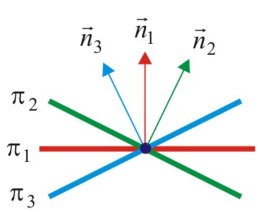
The normal vectors are not coplanar. Indeed:

$$\begin{matrix} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \vec{n}_2 \times \vec{n}_3 = & 2 & -5 & 1 & 2 & -5 \\ & -3 & 6 & 2 & -3 & 6 \end{matrix} = (-16,-7,-3)$$

$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = (1,-3,-2) \cdot (-16,-7,-3) = -16 + 21 + 6 = 11 \neq 0$

C Infinite Number of Solutions (I)
(Line Intersection – Non Parallel Planes and Coplanar Normal Vectors)





$L = \pi_1 \cap \pi_2 \cap \pi_3$

Ex 2. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y + 2z = -2 & (1) \\ 3x - y + 14z = 6 & (2) \\ x + 2y = -5 & (3) \Rightarrow x = -5 - 2y \quad (4) \end{cases} \Rightarrow$$

$$\begin{cases} (-5 - 2y) + y + 2z = -2 & (5) \\ 3(-5 - 2y) - y + 14z = 6 & (6) \\ x = -5 - 2y & (4) \end{cases} \Rightarrow \begin{cases} -y + 2z = 3 & (5) \\ -7y + 14z = 21 & (6) \\ x = -5 - 2y & (4) \end{cases}$$

Note that (5) and (6) are equivalent (by multiplying (5) by 7 you get (6)).

Let chose $y = t$. Then:

(4) $\Rightarrow x = -5 - 2t$
(5) $\Rightarrow z = (3 + t)/2$

In this case:

- ⇨ The planes are *not parallel* but their normal vectors are *coplanar*: $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$.
- ⇨ The intersection is a *line*.
- ⇨ One scalar equation is a *combination* of the other two equations.
- ⇨ By solving the system (*), you may express two variables in terms of the third one using two equations.

The solution may be written as:

$$\begin{cases} x = -5 - 2t \\ y = t \\ z = (3+t)/2 \end{cases}$$

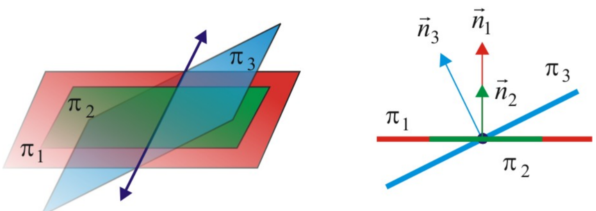
and represents the parametric equations of a line.
The normal vectors are coplanar. Indeed:

$$\begin{array}{cccccccc} & & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{n}_1 \times \vec{n}_3 & = & 1 & 1 & 2 & 1 & 1 & = (-4,2,1) \\ & & 1 & 2 & 0 & 1 & 2 & \end{array}$$

$$\vec{n}_2 \cdot (\vec{n}_1 \times \vec{n}_3) = (3,-1,14) \cdot (-4,2,1) = -12 - 2 + 14 = 0$$

D Infinite Number of Solutions (II)
(Line Intersection – Two Coincident Planes and one Intersecting Plane)

In this case:



$L = \pi_1 \cap \pi_2 \cap \pi_3$

- ⇨ *Two* planes are *coincident* and the third plane is *not parallel* to the coincident planes.
- ⇨ The intersection is a *line*.
- ⇨ Two scalar equations are *equivalent*. The coefficients A, B, C, D are *proportional* for these two equations.
- ⇨ You may express two variables in terms of the third one using two non equivalent equations.

Ex 3. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y - z = 2 & (1) \\ x - 2y + z = 4 & (2) \\ 2x - 4y + 2z = 8 & (3) \end{cases}$$

By multiplying (2) by 2, you get (3). The planes π_2 and π_3 are coincident. We may drop the equation (3).
Let isolate z from (1) and substitute in (2):

$$\begin{cases} z = x + y - 2 & (4) \\ x - 2y + (x + y - 2) = 4 & (5) \end{cases} \Rightarrow \begin{cases} z = x + y - 2 & (4) \\ 2x - y = 6 & (5) \end{cases}$$

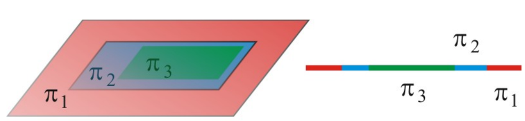
Let chose $x = t$. From (5) $y = -6 + 2x = -6 + 2t$, and from (4) $z = t - 6 + 2t - 2 = -8 + 3t$.
There are an infinite number of solutions given by:

$$L: \begin{cases} x = t \\ y = -6 + 2t \\ z = -8 + 3t \end{cases}$$

These are the parametric equations of the line of intersection of the three planes.
The coefficients A, B, C, D are proportional for equations (2) and (3) (coincident planes). The third plane with equation (1) intersects these coincident planes into a line.

E Infinite Number of Solutions (III)
(Plane Intersection – Three Coincident Planes)

In this case:



$\pi_1 \cap \pi_2 \cap \pi_3 = \pi_1 = \pi_2 = \pi_3$

- ⇨ The coefficients A, B, C, D are *proportional* for all three equations.
- ⇨ Any point of one plane is also a point on the other two planes.
- ⇨ The intersection is a *plane*.

Ex 4. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x - y - 2z = 1 \\ 2x - 2y - 4z = 2 \\ -4x + 4y + 8z = -4 \end{cases}$$

$$\begin{cases} x - y - 2z = 1 & (1) \\ 2x - 2y - 4z = 2 & (2) \\ -4x + 4y + 8z = -4 & (3) \end{cases} \Rightarrow \begin{array}{l} (2) = (1) \times 2 \\ (3) = (1) \times (-4) \end{array}$$

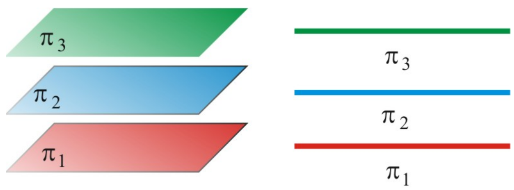
There are an infinite number of solutions given by:

$$\begin{cases} x = s + 2t + 1 \\ y = s \\ z = t \end{cases} \quad ; \quad s, t \in R$$

The three equations represent three coincident planes. There intersection is given by any of these planes.

F No Solution
(Parallel and Distinct Planes)

In this case:



- ⇨ There are three *parallel* and *distinct* planes.
- ⇨ There is *no point of intersection*.
- ⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).
- ⇨ The coefficients A, B, C are *proportional* but the coefficients A, B, C, D are *not proportional*.
- ⇨ By solving the system (*) you get *false* statements (like $0 = 1$).

Ex 5. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + 2y + 3z = 1 & (1) \\ 2x + 4y + 6z = -1 & (2) \\ -x - 2y - 3z = 3 & (3) \end{cases}$$

Let isolate x from (1) and substitute in (2) and (3):

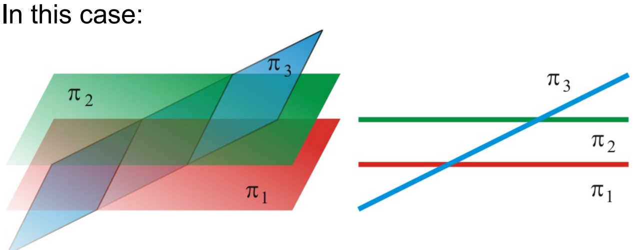
$$\begin{cases} x = 1 - 2y - 3z & (4) \\ 2(1 - 2y - 3z) + 4y + 6z = -1 & (5) \\ -(1 - 2y - 3z) - 2y - 3z = 3 & (6) \end{cases}$$

$$\begin{cases} x = 1 - 2y - 3z & (4) \\ 2 = -1 & (5) \Rightarrow \mathit{false} \\ -1 = 3 & (6) \Rightarrow \mathit{false} \end{cases}$$

There are no solutions and therefore no points of intersection.
Note that the coefficients A, B, C are proportional but the coefficients A, B, C, D are not proportional. So, the planes defined by the three equations are parallel and distinct.

G No Solution (II)
(H Configuration)

In this case:



- ⇨ Two planes are *parallel and distinct* and the third plane is *intersecting*.
- ⇨ There is *no point of intersection*.
- ⇨ The coefficients A, B, C are *proportional* for two planes.
- ⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).
- ⇨ By solving the system (*) you get false statements (like $0 = 1$).

Ex 6. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y - z = 1 & (1) \\ x + y + z = 2 & (2) \\ -2x - 2y + 2z = 3 & (3) \end{cases}$$

Let isolate z from (1) and substitute in (2) and (3):

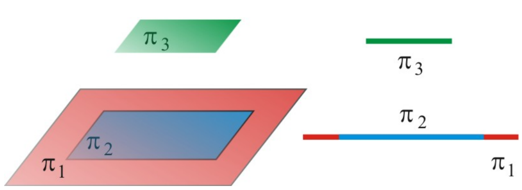
$$\begin{cases} z = x + y - 1 & (4) \\ x + y + (x + y - 1) = 2 & (5) \\ -2x - 2y + 2(x + y - 1) = 3 & (6) \end{cases}$$

$$\begin{cases} z = x + y - 1 & (4) \\ 2x + 2y = 3 & (5) \\ -2 = 3 & (6) \Rightarrow \mathit{false} \end{cases}$$

There is no solution and therefore no point of intersection.
The planes (1) and (3) are parallel but distinct. The plane (2) is not parallel to the planes (1) and (3).
The planes are in the H configuration.

H No Solution (III)

In this case:



- ⇨ Three planes are *parallel* but only *two are coincident*.

Ex 7. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + 2y + 3z = 1 & (1) \\ 3x + 6y + 9z = 3 & (2) \\ -2x - 4y - 6z = 2 & (3) \end{cases}$$

Let isolate x from (1) and substitute in (2) and (3):

$$\begin{cases} x = 1 - 2y - 3z & (4) \\ 3(1 - 2y - 3z) + 6y + 9z = 3 & (5) \\ -2(1 - 2y - 3z) - 4y - 6z = 2 & (6) \end{cases}$$

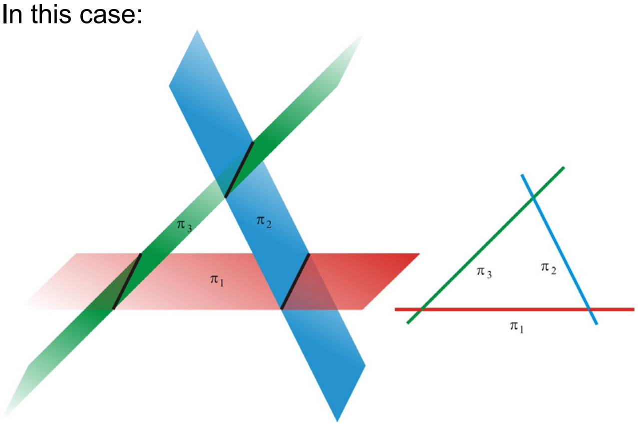
⇨ The coefficients A, B, C are *proportional* for all equations but the coefficients A, B, C, D are *proportional* only for two planes.
⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).
⇨ By solving the system (*) you get *false* statements (like $0 = 1$).

$$\begin{cases} x = 1 - 2y - 3z & (4) \\ 0 = 0 & (5) \\ -2 = 2 & (6) \Rightarrow \mathit{false} \end{cases}$$

There is no solution and therefore no point of intersection. All planes are parallel but only the planes (1) and (2) are coincident.

I No Solution (IV)
(Delta Configuration)

In this case:



⇨ The planes are *not parallel* (the coefficients A, B, C are *not proportional*).
⇨ The normal vectors are *coplanar* ($\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$).
⇨ There is *no point of intersection* between all three planes.
⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).
⇨ By solving the system (*) you get *false* statements (like $0 = 1$).

Ex 8. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} 2x + y + z = 1 & (1) \\ -x + y + z = -1 & (2) \\ x + y + z = 0 & (3) \end{cases}$$

Let isolate z from (1) and substitute in (2) and (3):

$$\begin{cases} z = 1 - 2x - y & (4) \\ -x + y + (1 - 2x - y) = -1 & (5) \\ x + y + (1 - 2x - y) = 0 & (6) \end{cases}$$

$$\begin{cases} z = 1 - 2x - y & (4) \\ -3x = -2 & (5) \Rightarrow x = 2/3 \\ -x = -1 & (6) \Rightarrow x = 1 \end{cases}$$

The equations (5) and (6) are incompatible.
There is no solution and therefore no point of intersection.
The coefficients A, B, C are not proportional and therefore the planes are not parallel.
The normal vectors are coplanar. Indeed:

$$\vec{n}_1 = (2,1,1); \quad \vec{n}_2 = (-1,1,1); \quad \vec{n}_3 = (1,1,1)$$

$$\begin{array}{ccccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{n}_2 \times \vec{n}_3 = & -1 & 1 & 1 & -1 & 1 & = (0,2,-2) \\ & 1 & 1 & 1 & 1 & 1 & \end{array}$$

$$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0 + 2 - 2 = 0$$

The planes are in the Delta configuration.

Reading: Nelson Textbook, Pages 520-529
Homework: Nelson Textbook: Page 530 #8, 9, 10, 13