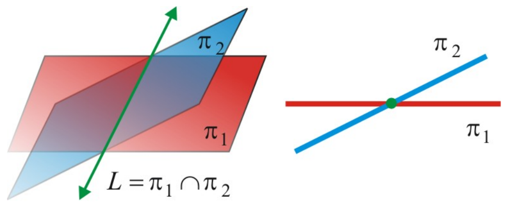
9.3 Intersection of two Planes

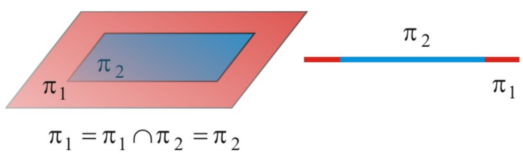
A Relative Position of two Planes

Two planes may be:

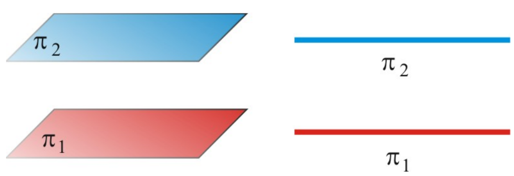
a) *intersecting* (into a line)



b) *coincident*



c) *distinct*



$\pi_1 \cap \pi_2 = \varnothing$

B Intersection of two Planes

Let consider two plane given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$
$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \quad (*)$$

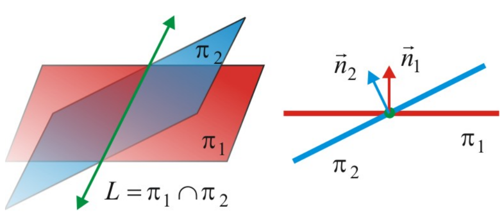
There are *two* equations are *three* unknowns.

Notes:

1. A normal vector to the plane π_1 is $\vec{n}_1 = (A_1, B_1, C_1)$ and a normal vector to the plane π_2 is $\vec{n}_2 = (A_2, B_2, C_2)$.
2. If the planes are *parallel* then coefficients A, B, C are *proportional*.
3. If the planes are *coincident* then coefficients A, B, C, D are *proportional*.
4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution*.

C Non Parallel Planes (Line Intersection)

In this case:



- ⇨ The coefficients A, B, C in the scalar equations are *not proportional*.
- ⇨ The normal vectors are *not parallel*: $\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$.
- ⇨ By solving the system (*) you will be able to find two variables in terms of the third variable.
- ⇨ There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- ⇨ The intersection is a *line* and a *direction vector* for this line is $\vec{u} = \vec{n}_1 \times \vec{n}_2$.

Ex 1. Find the point(s) of intersection of the following two planes. Give a geometric representation of the solution(s).

$\pi_1 : -2x + 3y + z + 6 = 0$

$\pi_2 : 3x - y + 2z - 2 = 0$

$$\begin{cases} -2x + 3y + z + 6 = 0 & (1) \\ 3x - y + 2z - 2 = 0 & (2) \end{cases}$$

Isolate z from (1) and substitute into (2):

$$\begin{cases} z = 2x - 3y - 6 & (3) \\ 3x - y + 2(2x - 3y - 6) - 2 = 0 & (4) \end{cases}$$

$$\begin{cases} z = 2x - 3y - 6 & (3) \\ 7x - 7y - 14 = 0 & (4) \end{cases}$$

Let chose $x = t$. Then: $(4) \Rightarrow y = x - 2 = t - 2$
$(3) \Rightarrow z = 2x - 3(x - 2) - 6 = -x = -t$

For any $t \in R$ there are $x, y, z \in R$ such that $x = t$, $y = t - 2$, and $z = -t$ satisfy the system of equations. There are an infinite number of solutions and therefore an infinite number of points of intersections given by the line with the parametric equations:

$$L = \pi_1 \cap \pi_2 : \begin{cases} x = t \\ y = t - 2 \\ z = -t \end{cases}$$

Note that the normal vectors $\vec{n}_1 = (-2, 3, 1)$ and $\vec{n}_2 = (3, -1, 2)$ are not parallel.

D Coincident Planes
(Plane Intersection)

In this case:

$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$

- ⇨ The planes are *parallel* and *coincident*.
- ⇨ The coefficients A, B, C, D in the scalar equations are *proportional*.
- ⇨ One equation in the system (*) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).
- ⇨ By solving the system of equations (*), you get a *true* statement (like $0 = 0$).
- ⇨ There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- ⇨ The intersection is a *plane*.

Ex 2. Find the point(s) of intersection of the following two planes. Give a geometric representation of the solution(s).

$\pi_1 : x - 2y + 3z + 1 = 0$

$\pi_2 : -3x + 6y - 9z - 3 = 0$

$$\begin{cases} x - 2y + 3z + 1 = 0 & (1) \\ -3x + 6y - 9z - 3 = 0 & (2) \end{cases}$$

Let solve (1) for x and substitute into (2).

$$\begin{cases} x = 2y - 3z - 1 & (3) \\ -3(2y - 3z - 1) + 6y - 9z - 3 = 0 & (4) \end{cases}$$

$$\begin{cases} x = 2y - 3z - 1 & (3) \\ -6y + 9z + 3 + 6y - 9z - 3 = 0 & (4) \end{cases}$$

$$\begin{cases} x = 2y - 3z - 1 & (3) \\ 0 = 0 & (4) \quad (true) \end{cases}$$

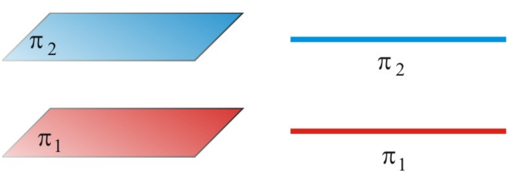
For any $y \in R$ and any $z \in R$ there is an $x \in R$ given by: $x = 2y - 3z - 1$ satisfying the system. There are an infinite number of solutions and therefore an infinite number of points of intersection. The intersection is the plane:

$\pi = \pi_1 \cap \pi_2 : x = 2y - 3z - 1$

Note that by multiplying the equation (1) by -3 you get the equation (2). The coefficients A, B, C, D are proportional.

E Parallel and Distinct Planes
(No Intersection)

In this case:



- ⇨ The planes are *parallel* and *distinct*.
- ⇨ The coefficients A, B, C in the scalar equations are *proportional* but the coefficients A, B, C, D are *not proportional*.
- ⇨ By solving the system (*) you get a *false* statement (like $0 = 1$).
- ⇨ There is *no solution* and therefore *no point of intersection* between the two planes.

Ex 3. Find the point(s) of intersection of the following two planes. Give a geometric representation of the solution(s).

$\pi_1 : x - y - 2z + 1 = 0$

$\pi_2 : -4x + 4y + 8z - 3 = 0$

$$\begin{cases} x - y - 2z + 1 = 0 & (1) \\ -4x + 4y + 8z - 3 = 0 & (2) \end{cases}$$

Let isolate y from (1) and substitute into (2).

$$\begin{cases} y = x - 2z + 1 & (3) \\ -4x + 4(x - 2z + 1) + 8z - 3 = 0 & (4) \end{cases}$$

$$\begin{cases} y = x - 2z + 1 & (3) \\ -4x + 4x - 8z + 4 + 8z - 3 = 0 & (4) \end{cases}$$

$$\begin{cases} y = x - 2z + 1 & (3) \\ 1 = 0 & (4) \quad (false) \end{cases}$$

The system of equations does not have any solution and therefore there is no point of intersection between the planes. The planes are parallel and distinct.
Note that the coefficients A, B, C are proportional but the coefficients A, B, C, D are not proportional.

Ex 4. Classify each pair of planes as distinct, coincident, or intersecting. Do not attempt to solve algebraically the system of equations.

a) $\pi_1 : 2x - 3y + z - 1 = 0$, $\pi_2 : 4x - 6y + 2z - 2 = 0$
The coefficients A, B, C, D are proportional.
Therefore the planes are coincident.

b) $\pi_1 : 3x + 6y - 9z - 3 = 0$, $\pi_2 : 2x + 4y - 6z - 4 = 0$
The coefficients A, B, C are proportional but the coefficients A, B, C, D are not proportional. Therefore the planes are parallel and distinct.

c) $\pi_1 : x + 2y + 3z + 1 = 0$, $\pi_2 : 3x + 2y + z + 2 = 0$
The coefficients A, B, C are not proportional. Therefore the planes are intersecting into a line.

Reading: Nelson Textbook, Pages 510-515
Homework: Nelson Textbook: Page 515 # 6abc, 8, 10, 11, 12