### A Normal Equation of a Plane

A plane may be determined by a **point** \( P_0(x_0, y_0, z_0) \) and a **vector** perpendicular to the plane \( \vec{n} \) called the **normal vector**.

If \( P(x, y, z) \) is a generic point on the plane, then: 
\[
\overrightarrow{P_0P} \perp \vec{n} \quad \text{and:} \\
\overrightarrow{P_0P} \cdot \vec{n} = 0
\]

This is the **normal equation** of a plane.

### B Cartesian Equation of a Plane

Let write the normal vector of a plane in the form: 
\[
\vec{n} = (A, B, C)
\]

Then, the normal equation (1) may be written as:
\[
(x-x_0, y-y_0, z-z_0) \cdot (A, B, C) = 0 \\
Ax + By + Cz = Ax_0 + By_0 + Cz_0
\]

or:
\[
Ax + By + Cz + D = 0 \quad (2)
\]
equation which is called the **Cartesian equation** of a plane.

Note. A normal vector to the plane is:
\[
\vec{n} = \vec{u} \times \vec{v}
\]
where \( \vec{u} \) and \( \vec{v} \) are the direction vectors of the plane.

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**Ex 1.** Consider the plane \( \pi \) defined the Cartesian equation \( 2x - 3y + 6z + 12 = 0 \).

a) Find a normal vector to this plane.
\[
\vec{n} = (2, -3, 6)
\]

b) Find two points on this plane.
If \( x = 0 \) and \( y = 0 \) then \( z = -2 \). So \( (0, 0, -2) \in \pi \).
If \( x = 0 \) and \( z = 0 \) then \( y = 4 \). So \( (0, 4, 0) \in \pi \).

c) Find if the point \( P(1, 2, 3) \) is a point on this plane.
\[
2(1) - 3(2) + 6(3) + 12 = 26 \neq 0 \Rightarrow P \notin \pi
\]

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**Ex 2.** Find the Cartesian equation of a plane \( \pi \) that passes through the points \( A(1, -1, 0) \), \( B(0, 0, 1) \), and \( C(0, -2, 1) \).

\[
\vec{u} = \overrightarrow{AB} = (-1, 1, 1); \quad \vec{v} = \overrightarrow{AC} = (-1, -1, 1) \]
\[
\vec{n} = \vec{u} \times \vec{v} = -1 \quad 1 \quad 1 \quad -1 -1 1 = (2, 0, 2) = (A, B, C)
\]
\[
2x + 2z + D = 0 \\
A(1, -1, 0) \in \pi \Rightarrow 2(1) + 2(0) + D = 0 \Rightarrow D = -2
\]
\[
\therefore 2x + 2z - 2 = 0 \text{ or } x + z = 1 = 0
\]

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**Ex 3.** Find parametric and vector equations for the plane:
\[
\pi: x - 2y + 3z - 6 = 0
\]

**Method #1**
\[
\begin{align*}
\begin{cases}
x = 6 + 2s - 3t \\
y = s \\
z = t
\end{cases} \\
\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-3, 0, 1), \quad s, t \in \mathbb{R}
\end{align*}
\]

**Method #2**
\[
\begin{align*}
A(0, 0, 2) \in \pi; \quad B(0, -3, 0) \in \pi; \quad C(6, 0, 0) \in \pi \\
\vec{u} = \overrightarrow{AB} = (0, -3, -2); \quad \vec{v} = \overrightarrow{AC} = (6, 0, -2) \\
\therefore \vec{r} = (0, 0, 2) + s(0, -3, -2) + t(6, 0, -2); \quad s, t \in \mathbb{R} \\
x = 6t \\
y = -3s \\
z = 2 - 2s - 2t
\end{align*}
\]

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**Ex 4.** Find the intersections with the coordinate axes for the plane \( \pi: 3x + 2y + z - 6 = 0 \). Represent the plane graphically.

Let \( A = \pi \cap x\text{-axis} \). \( y_A = z_A = 0 \Rightarrow x_A = 2 \).
\[
\therefore x - \text{int} = A(2, 0, 0).
\]

Let \( B = \pi \cap y\text{-axis} \). \( x_B = z_B = 0 \Rightarrow y_B = 3 \).
\[
\therefore y - \text{int} = B(0, 3, 0).
\]

Let \( C = \pi \cap z\text{-axis} \). \( x_C = y_C = 0 \Rightarrow z_C = 6 \).
\[
\therefore z - \text{int} = C(0, 0, 6).
\]
### Ex 5. Find the Cartesian equation of a plane with \( x - \text{int} = -1 \), \( y - \text{int} = 2 \), and \( z - \text{int} = -3 \).

\[
Ax + By + Cz + D = 0
\]

\[
x - \text{int} = -\frac{D}{A} = -1 \Rightarrow A = D
\]

\[
y - \text{int} = -\frac{D}{B} = 2 \Rightarrow B = -\frac{D}{2}
\]

\[
z - \text{int} = -\frac{D}{C} = -3 \Rightarrow C = \frac{D}{3}
\]

\[
Dx - \frac{D}{2}y + \frac{D}{3}z + D = 0
\]

\[
\therefore x - \frac{y}{2} + \frac{z}{3} + 1 = 0 \text{ or } 6x - 3y + 2z + 6 = 0
\]

### F Angle between two Planes

The angle between two plane s is defined as the angle between their normal vectors:

\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}
\]

Note. Using this formula, you may get an acute or an obtuse angle depending on the normal vectors which are used.

#### Ex 6. Find the angle between each pair of planes.

**a)** \( \pi_1 : x + 2y + 3z + 1 = 0 \), \( \pi_2 : 3x + 2y + z + 2 = 0 \)

\( \vec{n}_1 = (1,2,3); \quad \vec{n}_2 = (3,2,1) \)

\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(1)(3) + (2)(2) + (3)(1)}{\sqrt{1^2 + 2^2 + 3^2 \sqrt{3^2 + 2^2 + 1^2}}} = \frac{10}{14}\]

\[
\therefore \theta = \cos^{-1}\left(\frac{10}{14}\right) \approx 44.42^\circ
\]

**b)** \( \pi_1 : x + y + z + 1 = 0 \), \( \pi_2 : x - y - 1 = 0 \)

\( \vec{n}_1 = (1,1,1); \quad \vec{n}_2 = (1,-1,0) \)

\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(1)(1) + (1)(-1) + (1)(0)}{\sqrt{1^2 + 1^2 + 1^2 \sqrt{1^2 + (-1)^2 + 0^2}}} = \frac{0}{\sqrt{3}} = 0
\]

\[
\therefore \theta = 90^\circ
\]

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**Reading:** Nelson Textbook, Pages 461-468

**Homework:** Nelson Textbook: Page 468 #1, 5, 7, 8, 9a, 11, 13, 17