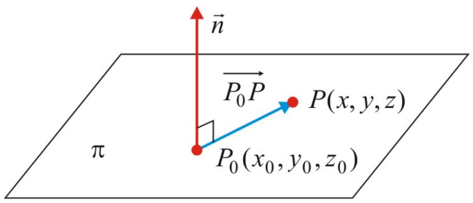
8.5 Cartesian Equation of a Plane

A Normal Equation of a Plane
A plane may be determined by a *point* $P_0(x_0, y_0, z_0)$ and a *vector* perpendicular to the plane $\vec{n}$ called the *normal vector*.



If $P(x, y, z)$ is a generic point on the plane, then:
$\overrightarrow{P_0P} \perp \vec{n}$ and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \quad (1)$$

This is the *normal equation* of a plane.

B Cartesian Equation of a Plane
Let write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation (1) may be written as:

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0$$

$$Ax + By + Cz - Ax_0 - By_0 - Cz_0 = 0$$

or:

$$Ax + By + Cz + D = 0 \quad (2)$$

equation which is called the *Cartesian equation* of a plane.

Note. A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where $\vec{u}$ and $\vec{v}$ are the direction vectors of the plane.

Ex 1. Consider the plane π defined the Cartesian equation $\pi : 2x - 3y + 6z + 12 = 0$.

a) Find a normal vector to this plane.
$\vec{n} = (2, -3, 6)$

b) Find two points on this plane.
If $x = 0$ and $y = 0$ then $z = -2$. So $(0,0,-2) \in \pi$.
If $x = 0$ and $z = 0$ then $y = 4$. So $(0,4,0) \in \pi$.

c) Find if the point $P(1,2,3)$ is a point on this plane.
$2(1) - 3(2) + 6(3) + 12 = 26 \neq 0 \Rightarrow \therefore P \notin \pi$

Ex 2. Find the Cartesian equation of a plane π that passes through the points $A(1,-1,0)$, $B(0,0,1)$, and $C(0,-2,1)$.

$\vec{u} = \overrightarrow{AB} = (-1,1,1); \quad \vec{v} = \overrightarrow{AC} = (-1,-1,1)$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{matrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 \end{matrix} = (2,0,2) = (A, B, C)$$

$2x + 2z + D = 0$
$A(1,-1,0) \in \pi \Rightarrow 2(1) + 2(0) + D = 0 \Rightarrow D = -2$
$\therefore 2x + 2z - 2 = 0 \text{ or } x + z - 1 = 0$

Ex 3. Find parametric and vector equations for the plane:
$\pi : x - 2y + 3z - 6 = 0$

Method #1

$$\begin{cases} x = 6 + 2s - 3t \\ y = s \qquad ; \quad s, t \in R \\ z = t \end{cases}$$

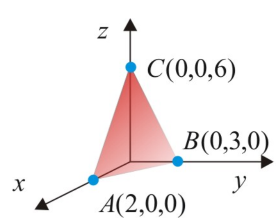
$\vec{r} = (6,0,0) + s(2,1,0) + t(-3,0,1), \quad s, t \in R$

Method #2
$A(0,0,2) \in \pi; B(0,-3,0) \in \pi; C(6,0,0) \in \pi$
$\vec{u} = \overrightarrow{AB} = (0,-3,-2); \quad \vec{v} = \overrightarrow{AC} = (6,0,-2)$
$\therefore \vec{r} = (0,0,2) + s(0,-3,-2) + t(6,0,-2); \quad s, t \in R$

$$\begin{cases} x = 6t \\ y = -3s \\ z = 2 - 2s - 2t \end{cases}$$

Ex 4. Find the intersections with the coordinate axes for the plane $\pi : 3x + 2y + z - 6 = 0$. Represent the plane graphically.

Let $A = \pi \cap x-axis$. $y_A = z_A = 0 \Rightarrow x_A = 2$.
$\therefore x - \text{int} = A(2,0,0)$.
Let $B = \pi \cap y-axis$. $x_B = z_B = 0 \Rightarrow y_B = 3$.
$\therefore y - \text{int} = B(0,3,0)$.
Let $C = \pi \cap z-axis$. $x_C = y_C = 0 \Rightarrow z_C = 6$.
$\therefore z - \text{int} = C(0,0,6)$.



Ex 5. Find the Cartesian equation of a plane with $x-\text{int}=-1$, $y-\text{int}=2$, and $z-\text{int}=-3$.

$$Ax+By+Cz+D=0$$

$$x-\text{int}=-\frac{D}{A}=-1 \Rightarrow A=D$$

$$y-\text{int}=-\frac{D}{B}=2 \Rightarrow B=-\frac{D}{2}$$

$$z-\text{int}=-\frac{D}{C}=-3 \Rightarrow C=\frac{D}{3}$$

$$Dx-\frac{D}{2}y+\frac{D}{3}z+D=0$$

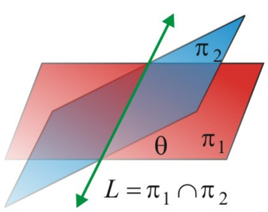
$$\therefore x-\frac{y}{2}+\frac{z}{3}+1=0 \; or \; \therefore 6x-3y+2z+6=0$$

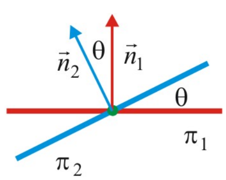
F Angle between two Planes

The *angle* between two plane s is defined as the angle between their *normal vectors*:

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Note. Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.





Ex 6. Find the angle between each pair of planes.

a) $\pi_1: x+2y+3z+1=0$, $\pi_2: 3x+2y+z+2=0$

$$\vec{n}_1=(1,2,3); \quad \vec{n}_2=(3,2,1)$$

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}=\frac{(1)(3)+(2)(2)+(3)(1)}{\sqrt{1^2+2^2+3^2}\sqrt{3^2+2^2+1^2}}=\frac{10}{14}$$

$$\therefore \theta=\cos^{-1}(10/14)\cong 44.42°$$

b) $\pi_1: x+y+z+1=0$, $\pi_2: x-y-1=0$

$$\vec{n}_1=(1,1,1); \quad \vec{n}_2=(1,-1,0)$$

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}=\frac{(1)(1)+(1)(-1)+(1)(0)}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-1)^2+0^2}}=\frac{0}{\sqrt{3}\sqrt{2}}=0$$

$$\therefore \theta=90°$$

Reading: Nelson Textbook, Pages 461-468
Homework: Nelson Textbook: Page 468 #1, 5, 7, 8, 9a, 11, 13, 17

