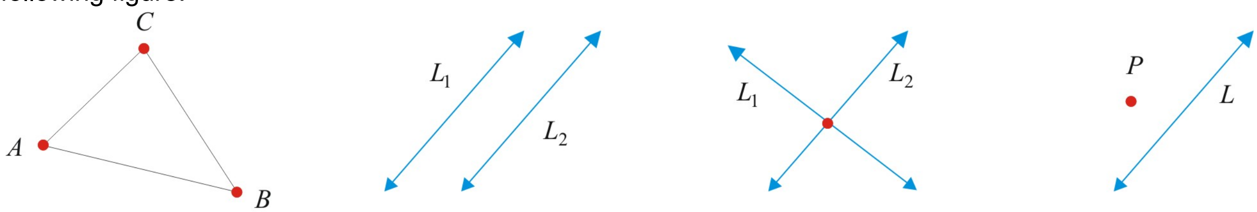
8.4 Vector and Parametric Equations of a Plane

A Planes

A plane may be determined by points and lines, There are four main possibilities as represented in the following figure:

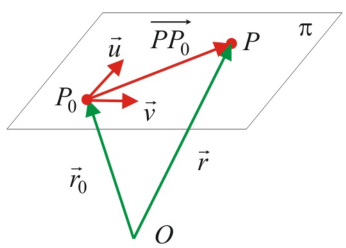


a) plane determined by three points
b) plane determined by two parallel lines
c) plane determined by two intersecting lines
d) plane determined by a line and a point

B Vector Equation of a Plane

Let consider a plane π.

Two vectors $\vec{u}$ and $\vec{v}$, parallel to the plane π but not parallel between them, are called *direction vectors* of the plane π.



The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point $P(x, y, z)$ of the plane is a *linear combination* of direction vectors $\vec{u}$ and $\vec{v}$:

$$\overrightarrow{P_0P} = s\vec{u} + t\vec{v}; \quad s,t \in R$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v}; \quad s,t \in R$$

Ex 1. A plane π is given by the following vector equation:

$\pi : \vec{r} = (-1,0,2) + s(0,0,1) + t(1,0,-1); \quad s,t \in R$

a) Find two points on this plane.

If $s = 0, t = 0$, then $\vec{r} = (-1,0,2) \Rightarrow P_0(-1,0,2) \in \pi$.

If $s = 1, t = 2$, then $\vec{r} = (-1,0,2) + (0,0,1) + 2(1,0,-1) = (1,0,1)$

$\therefore A(1,0,1) \in \pi$.

b) Find one line on this plane.

Let $L : \vec{r} = (-1,0,2) + s(0,0,1); \quad s \in R$.

$\therefore L \in \pi$

c) Find the vector equation of a line $L_\perp$ that passes through the origin and is perpendicular to this plane.

A direction vector for the line $L_\perp$ is:

$$\begin{matrix} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{u} \times \vec{v} = & 0 & 0 & 1 & 0 & 0 & = (0,1,0) \\ & 1 & 0 & -1 & 1 & 0 & \end{matrix}$$

$L_\perp : \vec{r} = q(0,1,0); \quad q \in R$

C Parametric Equations of a Plane

Let write vector equation of the plane as:

$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} ; \quad s,t \in R$$

These are the *parametric equations* of a line.

Ex 2. Convert the vector equation to the parametric equations.

$\vec{r} = (-1,0,2) + s(0,1,-1) + t(1,-2,0); \quad s,t \in R$

$(x, y, z) = (-1,0,2) + s(0,1,-1) + t(1,-2,0); \quad s,t \in R$

$$\therefore \begin{cases} x = -1 + t \\ y = s - 2t \\ z = 2 - s \end{cases} ; \quad s,t \in R$$

Ex 3. Convert the parametric equations to the vector equation.

$$\begin{cases} x = 1 + s - 2t \\ y = 3t \\ z = 4 - s \end{cases} ; \quad s,t \in R$$

$\therefore \vec{r} = (1,0,4) + s(1,0,-1) + t(-2,3,0); \quad s,t \in R$

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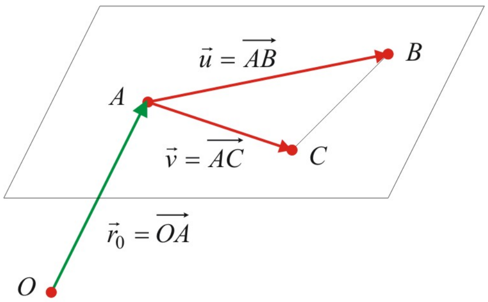
Ex 4. (Plane determined by three points)
Find the vector equation of the plane π that passes through the points $A(0,1,-1)$, $B(2,-1,0)$, and $C(0,0,1)$.



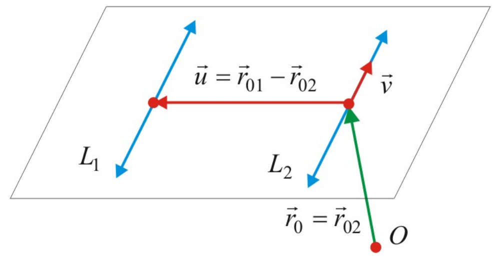
Let $\vec{r}_0 = \overrightarrow{OA} = (0,1,-1)$, $\vec{u} = \overrightarrow{AB} = (2,-2,1)$, and $\vec{v} = \overrightarrow{AC} = (0,-1,2)$. Then:

$\pi : \vec{r} = (0,1,-1) + s(2,-2,1) + t(0,-1,2); \quad s,t \in R$

Ex 5. (Plane determined by two parallel and distinct lines)
Find the vector and parametric equations of the plane π that contains the following parallel and distinct lines:

$L_1 : \vec{r} = (1,2,1) + s(0,-1,-2); \quad s \in R$

$L_2 : \vec{r} = (3,4,0) + t(0,1,2); \quad t \in R$



Let $\vec{r}_0 = \vec{r}_{02} = (3,4,0)$, $\vec{u} = \vec{r}_{01} - \vec{r}_{02} = (-2,-2,1)$, and $\vec{v} = \vec{u}_2 = (0,1,2)$. Then:

$\therefore \pi : \vec{r} = (3,4,0) + s(-2,-2,1) + t(0,1,2); \quad s,t \in R$

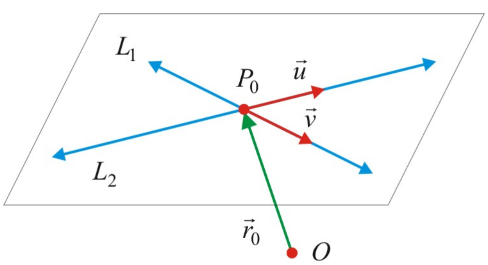
and

$$\therefore \pi : \begin{cases} x = 3 - 2s \\ y = 4 - 2s + t \\ z = s + 2t \end{cases} ; \quad s,t \in R$$

Ex 6. (Plane determined by two intersecting lines)
Find the vector equation of the plane π determined by the following intersecting lines.

$L_1 : \vec{r} = (0,0,1) + s(-1,0,0); \quad s \in R$

$L_2 : \vec{r} = (-3,0,1) + t(0,0,2); \quad t \in R$



Let first find the point of intersection.

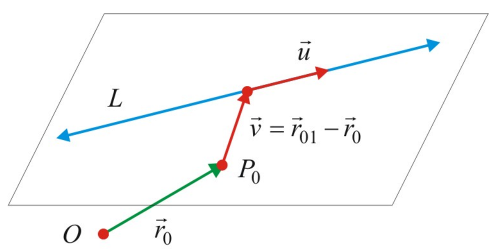
$$\begin{cases} -s = -3 \\ 0 = 0 \\ 1 = 1 + 2t \end{cases} \Rightarrow s = -3 \text{ and } t = 0$$

$P_0 = L_1 \cap L_2 \Rightarrow P_0 = (-3,0,1)$

Let $\vec{r}_0 = \overrightarrow{OP_0} = (-3,0,1)$, $\vec{u} = \vec{u}_2 = (0,0,2)$, and $\vec{v} = \vec{u}_1 = (-1,0,0)$. Then:

$\therefore \pi : \vec{r} = (-3,0,1) + s(0,0,2) + t(-1,0,0); \quad s,t \in R$

Ex 7. (Plane determined by a line and an external point)
Find the vector equation of the plane π that passes through the origin and contains the line
$L : \vec{r} = (0,1,2) + t(-1,0,3); \quad t \in R$.



Let $\vec{r}_0 = (0,0,0)$, $\vec{u} = (-1,0,3)$, and $\vec{v} = (0,1,2) - (0,0,0) = (0,1,2)$. Then the vector equation of the plane π is:

$\therefore \pi : \vec{r} = s(-1,0,3) + t(0,1,2); \quad s,t \in R$

Reading: Nelson Textbook, Pages 453-458
Homework: Nelson Textbook: Page 459 #1, 2, 4, 6b, 7, 9, 10, 15

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