8.4 Vector and Parametric Equations of a Plane

A Planes
A plane may be determined by points and lines. There are four main possibilities as represented in the following figure:

- a) plane determined by three points
- b) plane determined by two parallel lines
- c) plane determined by two intersecting lines
- d) plane determined by a line and a point

B Vector Equation of a Plane
Let consider a plane \( \pi \).

Two vectors \( \vec{u} \) and \( \vec{v} \), parallel to the plane \( \pi \) but not parallel between them, are called direction vectors of the plane \( \pi \).

The vector \( \overrightarrow{PP_0} \) from a specific point \((x_0, y_0, z_0)\) to a generic point \((x, y, z)\) of the plane is a linear combination of direction vectors \( \vec{u} \) and \( \vec{v} \):

\[
\overrightarrow{PP_0} = s\vec{u} + t\vec{v}; \quad s, t \in \mathbb{R}
\]

The vector equation of the plane is:

\[
\pi : \vec{r} = \overrightarrow{PP_0} + s\vec{u} + t\vec{v}; \quad s, t \in \mathbb{R}
\]

Ex 1. A plane \( \pi \) is given by the following vector equation:

\[
\pi : \vec{r} = (-1,0,2) + s(0,0,1) + t(1,0,-1); \quad s, t \in \mathbb{R}
\]

a) Find two points on this plane.

If \( s = 0, t = 0 \), then \( \vec{r} = (-1,0,2) \Rightarrow P_0(-1,0,2) \in \pi \).

If \( s = 1, t = 2 \), then \( \vec{r} = (-1,0,2) + (0,0,1) + 2(1,0,-1) = (1,0,1) \)

\( \therefore A(1,0,1) \in \pi \).

b) Find one line on this plane.

Let \( L : \vec{r} = (-1,0,2) + s(0,0,1); \quad s \in \mathbb{R} \).

\( \therefore L \in \pi \)

c) Find the vector equation of a line \( L_\perp \) that passes through the origin and is perpendicular to this plane.

A direction vector for the line \( L_\perp \) is:

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{vmatrix} = (0,1,0)
\]

\[
L_\perp : \vec{r} = q(0,1,0); \quad q \in \mathbb{R}
\]

C Parametric Equations of a Plane
Let write vector equation of the plane as:

\[
(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)
\]

or:

\[
\begin{cases}
x = x_0 + su_x + tv_x \\
y = y_0 + su_y + tv_y \\
z = z_0 + su_z + tv_z
\end{cases}; \quad s, t \in \mathbb{R}
\]

These are the parametric equations of a line.

Ex 2. Convert the vector equation to the parametric equations.

\[
\vec{r} = (-1,0,2) + s(0,1,-1) + t(1,-2,0); \quad s, t \in \mathbb{R}
\]

\[
(x, y, z) = (-1,0,2) + s(0,1,-1) + t(1,-2,0); \quad s, t \in \mathbb{R}
\]

\[
\begin{cases}
x = -1 + t \\
y = s - 2t \\
z = 2 - s
\end{cases}; \quad s, t \in \mathbb{R}
\]

Ex 3. Convert the parametric equations to the vector equation.

\[
\begin{cases}
x = 1 + s - 2t \\
y = 3t \\
z = 4 - s
\end{cases}; \quad s, t \in \mathbb{R}
\]

\( \therefore \vec{r} = (1,0,4) + s(1,0,-1) + t(-2,3,0); \quad s, t \in \mathbb{R} \)
Ex 4. (Plane determined by three points)
Find the vector equation of the plane \( \pi \) that passes through the points \( A(0,1,-1) \), \( B(2,-1,0) \), and \( C(0,0,1) \).

Let \( \overrightarrow{OA} = (0,1,-1) \), \( \overrightarrow{AB} = (2,-2,1) \), and \( \overrightarrow{AC} = (0,-1,2) \). Then:
\[ \pi : \overrightarrow{r} = (0,1,-1) + s(2,-2,1) + t(0,-1,2); \quad s, t \in \mathbb{R} \]

Ex 5. (Plane determined by two parallel and distinct lines)
Find the vector and parametric equations of the plane \( \pi \) that contains the following parallel and distinct lines:
\( L_1 : \overrightarrow{r} = (1,2,1) + s(0,-1,-2); \quad s \in \mathbb{R} \)
\( L_2 : \overrightarrow{r} = (3,4,0) + t(0,1,2); \quad t \in \mathbb{R} \)

Let \( \overrightarrow{r_0} = (3,4,0) \), \( \overrightarrow{u_2} = (0,-1,-2) \), and \( \overrightarrow{v} = 0(1,2) \). Then:
\[ \pi : \overrightarrow{r} = (3,4,0) + s(-2,2,1) + t(0,1,2); \quad s, t \in \mathbb{R} \]

and
\[ \begin{align*}
    x &= 3 - 2s \\
    y &= 4 - 2s + t \\
    z &= s + 2t
\end{align*} \]

Ex 6. (Plane determined by two intersecting lines)
Find the vector equation of the plane \( \pi \) determined by the following intersecting lines.
\( L_1 : \overrightarrow{r} = (0,1,1) + s(-1,0,0); \quad s \in \mathbb{R} \)
\( L_2 : \overrightarrow{r} = (-3,0,1) + t(0,0,2); \quad t \in \mathbb{R} \)

Let first find the point of intersection.
\[ \begin{align*}
    -s &= -3 \\
    0 &= 0 \\n    1 &= 1 + 2t
\end{align*} \]
\[ \Rightarrow s = -3 \text{ and } t = 0 \]
Let \( P_0 = L_1 \cap L_2 \Rightarrow P_0 = (-3,0,1) \)

Let \( \overrightarrow{P_0} = (-3,0,1) \), \( \overrightarrow{u} = (0,0,2) \), and \( \overrightarrow{v} = (1,0,0) \). Then:
\[ \pi : \overrightarrow{r} = (0,0,2) - 3(0,0,2) + t(1,0,0); \quad s, t \in \mathbb{R} \]

Ex 7. (Plane determined by a line and an external point)
Find the vector equation of the plane \( \pi \) that passes through the origin and contains the line
\( L : \overrightarrow{r} = (0,1,2) + t(-1,0,3); \quad t \in \mathbb{R} \)

Let \( \overrightarrow{r_0} = (0,0,0) \), \( \overrightarrow{u} = (-1,0,3) \), and \( \overrightarrow{v} = (0,1,2) \). Then the vector equation of the plane \( \pi \) is:
\[ \pi : \overrightarrow{r} = s(-1,0,3) + t(0,1,2); \quad s, t \in \mathbb{R} \]

Reading: Nelson Textbook, Pages 453-458
Homework: Nelson Textbook: Page 459 #1, 2, 4, 6b, 7, 9, 10, 15