8.3 Vector, Parametric, and Symmetric Equations of a Line in R^3

A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$$

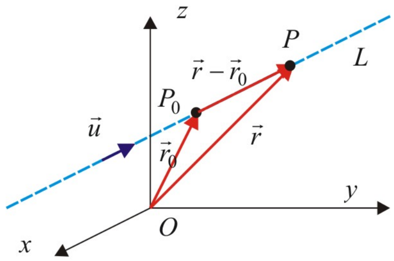
where:

⇨ $\vec{r} = \overrightarrow{OP}$ is the position vector of a *generic* point P on the line,

⇨ $\vec{r}_0 = \overrightarrow{OP_0}$ is the position vector of a *specific* point P_0 on the line,

⇨ $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line, and

⇨ t is a *real number* corresponding to the generic point P.



Ex 1. Find two vector equations of the line L that passes through the points $A(1,2,3)$ and $B(2,-1,0)$.

If we use the direction vector $\vec{u} = \overrightarrow{AB} = (1,-3,-3)$ and the point $A(1,2,3) \in L$, then the vector equation of the line L is:

$L: \vec{r} = (1,2,3) + t(1,-3,-3), \quad t \in R$

If we use the direction vector $\vec{u} = \overrightarrow{BA} = (-1,3,3)$ and the point $B(2,-1,0) \in L$, then the vector equation of the line L is:

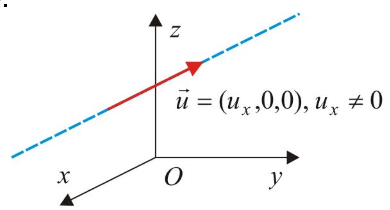
$L: \vec{r} = (2,-1,0) + s(-1,3,3), \quad s \in R$

Ex 2. Find the vector equation of a line L_2 that passes through the origin and is parallel to the line $L_1: \vec{r} = (-2,0,3) + t(-1,0,2), \quad t \in R$.

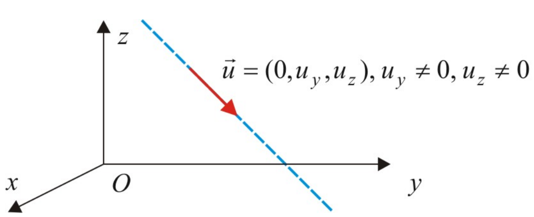
$\therefore L_2: \vec{r} = s(-1,0,2), \quad s \in R$

B Specific Lines

A line is *parallel to the x-axis* if $\vec{u} = (u_x,0,0), u_x \neq 0$. In this case, the line is also *perpendicular to the yz-plane*.



A line with $\vec{u} = (0,u_y,u_z), u_y \neq 0, u_z \neq 0$ is *parallel to the yz-plane*.



Ex 3. Find the vector equation of a line that:

a) passes through $A(3,-2,0)$ and is parallel to the y-axis

$\vec{r} = (3,-2,0) + t(0,1,0), \quad t \in R$

b) passes through $M(-1,0,4)$ and is perpendicular to the yz-plane

$\vec{r} = (-1,0,4) + t(1,0,0), \quad t \in R$

c) passes through $P(3,0,0)$ and is perpendicular to the x-axis

$\vec{r} = (3,0,0) + t(0,a,b), \quad t \in R$. At least one of a or b is not 0.

d) passes through the origin and is parallel to the xz-plane

$\vec{r} = t(a,0,b), \quad t \in R$. At least one of a or b is not 0.

C Parametric Equations

Let rewrite the vector equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$$

as:

$$(x,y,z) = (x_0,y_0,z_0) + t(u_x,u_y,u_z), \quad t \in R$$

The *parametric equations* of a line in R^3 are:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases}, \quad t \in R$$

Ex 4. Find the parametric equations of the line L that passes through the points $A(0,-1,2)$ and $B(1,-1,3)$. Describe the line.

$\vec{u} = \overrightarrow{AB} = (1,0,1); \quad A(0,-1,2) \in L$

$L: \vec{r} = (0,-1,2) + t(1,0,1), \quad t \in R$

$(x,y,z) = (0,-1,2) + t(1,0,1)$

$$L \therefore \begin{cases} x = t \\ y = -1 \\ z = 2 + t \end{cases}, \quad t \in R$$

The line is parallel to the xz-plane.

D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x - x_0 = tu_x \\ y - y_0 = tu_y \\ z - z_0 = tu_z \end{cases}, \quad t \in R$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$

$$u_x \neq 0, u_y \neq 0, u_z \neq 0$$

Ex 5. Convert the vector equation of the line $L: \vec{r} = (0,1,-3) + t(-1,2,0), \quad t \in R$ to the parametric and symmetric equations.

$$(x, y, z) = (0,1,-3) + t(-1,2,0)$$

$$\therefore \begin{cases} x = -t \\ y = 1 + 2t \\ z = -3 \end{cases}, \quad t \in R$$

$$\frac{x}{-1} = \frac{y - 1}{2} = \frac{z + 3}{0}$$

$$\therefore \frac{x}{-1} = \frac{y - 1}{2}, z = -3$$

Ex 6. Convert the symmetric equations for a line: $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{4}$ to the parametric and vector equations.

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{4} = t \Rightarrow \begin{cases} x - 2 = 3t \\ y + 1 = -2t \\ z = 4t \end{cases}$$

$$\therefore \begin{cases} x = 2 + 3t \\ y = -1 - 2t \\ z = 4t \end{cases}, \quad t \in R$$

$$\therefore \vec{r} = (2,-1,0) + t(3,-2,4), \quad t \in R$$

Ex 7. For each case, find if the given point lies on the given line.

a) $L: \vec{r} = (1,2,-3) + t(0,1,-2); \quad P(1,4,-7)$

$$(1,4,-7) = (1,2,-3) + t(0,1,-2)$$

$$(1,4,-7) - (1,2,-3) = t(0,1,-2)$$

$$(0,2,-4) = t(0,1,-2) \Rightarrow t = 2 \Rightarrow \therefore P \in L$$

b) $L: \begin{cases} x = -2 + 3t \\ y = -t \\ z = 5 \end{cases}; \quad P(0,1,5)$

$$\left.\begin{cases} 0 = -2 + 3t \Rightarrow t = 2/3 \\ 1 = -t \Rightarrow t = -1 \\ 5 = 5 \Rightarrow true \end{cases}\right\} \Rightarrow \therefore P \notin L$$

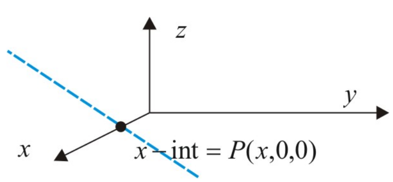
c) $L: \frac{x+1}{-2} = \frac{y-2}{1} = \frac{z}{-3}; \quad P(-3,3,-3)$

$$\frac{-3+1}{-2} = \frac{3-2}{1} = \frac{-3}{-3}$$

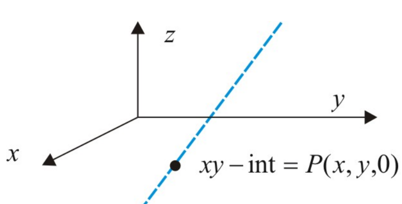
$$1 = 1 = 1 \Rightarrow P \in L$$

E Intersections

A line *intersects the x-axis* when $y = z = 0$.



A line *intersects the xy-plane* when $z = 0$.



Ex 7. Consider the line $L: \vec{r} = (3,-2,3) + t(-1,2,-3), \quad t \in R$. Find the intersection points between this line and the coordinates axes and planes.

$$L: \begin{cases} x = 3 - t \\ y = -2 + 2t \\ z = 3 - 3t \end{cases}$$

$$x = 0 \Rightarrow t = 3 \Rightarrow y = -2 + 2(3) = 4, z = 3 - 3(3) = -6$$

$$\therefore yz - \text{int} = A(0,4,-6) = L \cap yz - plane$$

$$y = 0 \Rightarrow t = 1 \Rightarrow x = 3 - 1 = 2, z = 3 - 3(1) = 0$$

$$\therefore xz - \text{int} = B(2,0,0) = L \cap xz - plane$$

$$z = 0 \Rightarrow t = 1 \Rightarrow x = 3 - 1 = 2, y = -2 + 2(1) = 0$$

$$\therefore xy - \text{int} = B(2,0,0) = L \cap xy - plane$$

$$\therefore x - \text{int} = B(2,0,0) = L \cap x - axis$$

Note that y-intercept and z-intercept do not exist.

Reading: Nelson Textbook, Pages 445-448

Homework: Nelson Textbook: Page 449 #1abc, 5acf, 6, 8, 9, 12, 13, 14

