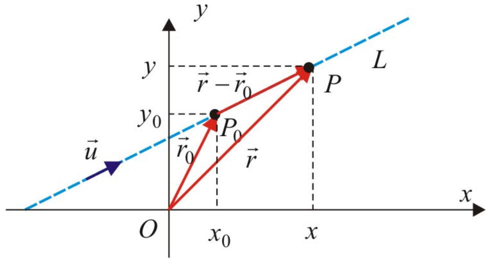
8.1 Vector and Parametric Equations of a Line in R^2

A Vector Equation of a Line in R^2

Let consider the line L that passes through the point $P_0(x_0, y_0)$ and is parallel to the vector $\vec{u}$.
The point $P(x, y)$ is a *generic point* on the line.



$$\overrightarrow{P_0P} = t\vec{u}$$
$$\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{u}$$
$$\vec{r} - \vec{r}_0 = t\vec{u}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$$

where:

- ⇨ $\vec{r} = \overrightarrow{OP}$ is the *position vector* of a *generic point* P on the line,
- ⇨ $\vec{r}_0 = \overrightarrow{OP_0}$ is the *position vector* of a *specific point* P_0 on the line,
- ⇨ $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line, and
- ⇨ t is a *real number* corresponding to the generic point P.

Note. The vector equation of a line is *not unique*. It depends on the specific point P_0 and on the direction vector $\vec{u}$ that are used.

Ex 1. The vector equation of the line L is given by:
$L: \vec{r} = (0,1) + t(-1,2), \quad t \in R$

a) Find a direction vector for this line.
$\therefore \vec{u} = (-1,2)$

b) Find a specific point on this line.
$\therefore P_0 = (0,1)$

c) Find the points A and B on this line corresponding to $t = 1$ and $t = 4$ respectively.
$t = 1 \Rightarrow \vec{r} = (0,1) + (-1,2) = (-1,3) \Rightarrow \therefore A(-1,3)$
$t = 4 \Rightarrow \vec{r} = (0,1) + 4(-1,2) = (-4,9) \Rightarrow \therefore B(-4,9)$

d) Explain what does represent the equation:
$\vec{r} = (0,1) + t(-1,2), \quad t \in [1,4]$
$\therefore$ The equation represents the set of points of the line segment AB, where $A(-1,3)$ and $B(-4,9)$.

e) Verify if the points $M(-2,5)$ and $N(2,3)$ are or not on the line. Hint: Try to find a t corresponding to each point.
$(-2,5) = (0,1) + t(-1,2) \Rightarrow (-2,5) - (0,1) = t(-1,2)$
$(-2,4) = t(-1,2) \Rightarrow t = 2 \Rightarrow \therefore M \in L$
$(2,3) = (0,1) + t(-1,2) \Rightarrow (2,3) - (0,1) = t(-1,2)$
$$(2,2) = t(-1,2) \Rightarrow \begin{cases} 2 = -t \Rightarrow t = -2 \\ 2 = 2t \Rightarrow t = 1 \end{cases} \Rightarrow \text{no solution}$$
$\therefore N \notin L$

Ex 2. Find two vector equations of the line L that passes through the points $A(2,-3)$ and $B(-1,2)$.

Let use $P_0 \equiv A(2,-3)$ and $\vec{u} = \overrightarrow{AB} = (-3,5)$. So:
$\therefore L: \vec{r} = (2,-3) + t(-3,5), \quad t \in R$
Let use $P_0 \equiv B(-1,2)$ and $\vec{u} = 2\overrightarrow{BA} = 2(3,-5) = (6,-10)$. So:
$\therefore L: \vec{r} = (-1,2) + s(6,-10), \quad s \in R$

B Parametric Equations of a Line in R^2

Let rewrite the vector equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y), \quad t \in R$$

Split this vector equation into the *parametric equations* of a line in R^2:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in R$$

Ex 3. Convert each vector equation into the parametric equations.

a) $\vec{r} = (1,-3) + t(-2,5), \quad t \in R$
$(x, y) = (1,-3) + t(-2,5), \quad t \in R$
$$\therefore \begin{cases} x = 1 - 2t \\ y = -3 + 5t \end{cases} \quad t \in R$$

b) $\vec{r} = (-2,0) + s(0,-3), \quad s \in R$
$(x, y) = (-2,0) + s(0,-3), \quad s \in R$
$$\therefore \begin{cases} x = -2 \\ y = -3s \end{cases} \quad s \in R$$

Ex 4. Convert the parametric equations of each line into a vector equation.

a) $\begin{cases} x = -2 + 3t \\ y = 5 - t \end{cases} \quad t \in R$

$P_0(-2,5)$ *and* $\vec{u} = (3,-1)$

$\therefore \vec{r} = (-2,5) + t(3,-1), \quad t \in R$

b) $\begin{cases} x = 1 - 3s \\ y = 2 \end{cases} \quad s \in R$

$P_0(1,2)$ *and* $\vec{u} = (-3,0)$

$\therefore \vec{r} = (1,2) + s(-3,0), \quad s \in R$

Ex 5. Find the points of intersection between the line given by $L : \vec{r} = (1,2) + t(-1,1), \quad t \in R$ and the coordinate axes.

$L : \begin{cases} x = 1 - t \\ y = 2 + t \end{cases} \quad t \in R$

Let $A = L \cap x - axis$ (the intersection with the x-axis). Then

$y_A = 0$ and $\begin{cases} x = 1 - t \\ 0 = 2 + t \Rightarrow t = -2 \end{cases} \Rightarrow x_A = 1 - (-2) = 3 \Rightarrow \therefore A(3,0)$

Let $B = L \cap y - axis$ (the intersection with the y-axis). Then

$x_B = 0$ and $\begin{cases} 0 = 1 - t \Rightarrow t = 1 \\ y = 2 + t \end{cases} \Rightarrow y_B = 2 + 1 = 3 \Rightarrow \therefore B(0,3)$

C Parallel Lines

Two lines L_1 and L_2 with direction vectors $\vec{u}_1$ and $\vec{u}_2$ are *parallel* ($L_1 \parallel L_2$) if:

$$\vec{u}_1 \parallel \vec{u}_2$$

or, there exists $k \in R$ such that:

$$\vec{u}_2 = k\vec{u}_1$$

or:

$$\vec{u}_1 \times \vec{u}_2 = \vec{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2y}}{u_{1y}} = k$$

Ex 6. Consider the line $L_1 : \vec{r} = (1,-3) + t(-1,2), \quad t \in R$. Find the vector equation of a line L_2, parallel to L_1 that passes through the point $M(-1,-13)$.

Because $L_1 \parallel L_2$, a valid direction vector for L_2 is $\vec{u}_1 = (-1,2)$. So, a possible vector equation for L_2 is:

$L_2 : \vec{r} = (-1,-13) + s(-1,2), \quad s \in R$

D Perpendicular Lines

Two lines L_1 and L_2 with direction vectors $\vec{u}_1$ and $\vec{u}_2$ are *perpendicular* ($L_1 \perp L_2$) if:

$$\vec{u}_2 \perp \vec{u}_1$$

or:

$$\vec{u}_1 \cdot \vec{u}_2 = \vec{0}$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

Ex 7. Show that $L_1 \perp L_2$ where:

$L_1 : \vec{r} = (-2,3) + t(2,-1), \quad t \in R$

$L_2 : \begin{cases} x = -2s \\ y = 5 - 4s \end{cases} \quad s \in R$

$\vec{u}_1 = (2,-1)$

$\vec{u}_2 = (-2,-4)$

$\vec{u}_1 \cdot \vec{u}_2 = (2)(-2) + (-1)(-4) = -4 + 4 = 0 \Rightarrow \therefore L_1 \perp L_2$

E 2D Perpendicular Vectors

Given a 2D vector $\vec{u} = (a,b)$, two 2D vectors perpendicular to $\vec{u}$ are $\vec{v} = (-b,a)$ and $\vec{w} = (b,-a)$.

Indeed:

$\vec{u} \cdot \vec{v} = (a,b) \cdot (-b,a) = -ab + ba = 0 \Rightarrow \vec{u} \perp \vec{v}$

Ex 8. Consider the line $L_1 : \vec{r} = (0,2) + t(2,-3), \quad t \in R$. Find the vector equation of a line L_2, perpendicular to L_1 that passes through the point $N(-3,0)$.

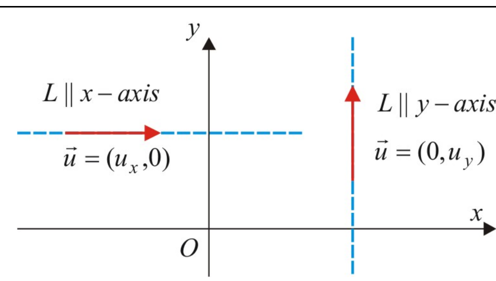
$\vec{u}_1 = (2,-3); \quad \vec{u}_2 \perp \vec{u}_1 \Rightarrow \vec{u}_2 = (3,2)$

$\therefore L_2 : \vec{r} = (-3,0) + s(3,2), \quad s \in R$

F Special Lines

A line *parallel to the x-axis* has a direction vector in the form $\vec{u} = (u_x,0), \quad u_x \neq 0$.

A line *parallel to the y-axis* has a direction vector in the form $\vec{u} = (0,u_y), \quad u_y \neq 0$.



Reading: Nelson Textbook, Pages 427-432

Homework: Nelson Textbook: Page 433 #2, 3, 4, 5, 8, 9, 10, 11, 13

