### A Vector Equation of a Line in R²

Let consider the line \( L \) that passes through the point \( P_0(x_0, y_0) \) and is parallel to the vector \( \vec{u} \). The point \( P(x, y) \) is a generic point on the line.

The vector equation of the line is:
\[
\vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in \mathbb{R}
\]

where:
- \( \vec{r} = \overrightarrow{OP} \) is the position vector of a generic point \( P \) on the line,
- \( \vec{r}_0 = \overrightarrow{O\vec{P}_0} \) is the position vector of a specific point \( P_0 \) on the line,
- \( \vec{u} \) is a vector parallel to the line called the direction vector of the line, and
- \( t \) is a real number corresponding to the generic point \( P \).

Note. The vector equation of a line is not unique. It depends on the specific point \( P_0 \) and on the direction vector \( \vec{u} \) that are used.

### B Parametric Equations of a Line in R²

Let rewrite the vector equation of a line:
\[
\vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in \mathbb{R}
\]
as:
\[
(x, y) = (x_0, y_0) + t(u_x, u_y), \quad t \in \mathbb{R}
\]

Split this vector equation into the parametric equations of a line in \( \mathbb{R}^2 \):
\[
\begin{align*}
x &= x_0 + tu_x, \quad t \in \mathbb{R} \\
y &= y_0 + tu_y, \quad t \in \mathbb{R}
\end{align*}
\]

### Ex 1

The vector equation of the line \( L \) is given by:
\[
L : \vec{r} = (0,1) + t(-1,2), \quad t \in \mathbb{R}
\]

- a) Find a direction vector for this line.
  \( \therefore \vec{u} = (-1,2) \)

- b) Find a specific point on this line.
  \( \therefore P_0 = (0,1) \)

- c) Find the points \( A \) and \( B \) on this line corresponding to \( t = 1 \) and \( t = 4 \) respectively.
  \( t = 1 \Rightarrow \vec{r} = (0,1) + (-1,2) = (-1,3) \Rightarrow \therefore A(-1,3) \)
  \( t = 4 \Rightarrow \vec{r} = (0,1) + 4(-1,2) = (-4,9) \Rightarrow \therefore B(-4,9) \)

- d) Explain what does represent the equation:
  \( \vec{r} = (0,1) + t(-1,2), \quad t \in [1,4] \)
  \( \therefore \) The equation represents the set of points of the line segment \( AB \), where \( A(-1,3) \) and \( B(-4,9) \).

- e) Verify if the points \( M(-2,5) \) and \( N(2,3) \) are or not on the line. Hint: Try to find a \( t \) corresponding to each point.
  \( (-2,5) = (0,1) + t(-1,2) \Rightarrow (-2,5) - (0,1) = t(-1,2) \)
  \( (-2,4) = t(-1,2) \Rightarrow t = 2 \Rightarrow \therefore M \in L \)
  \( (2,3) = (0,1) + t(-1,2) \Rightarrow (2,3) - (0,1) = t(-1,2) \)
  \( (2,2) = t(-1,2) \Rightarrow \begin{cases} 2 = -t \Rightarrow t = -2 \\ 2 = 2t \Rightarrow t = 1 \end{cases} \Rightarrow \text{no solution} \)
  \( \therefore N \notin L \)

### Ex 2

Find two vector equations of the line \( L \) that passes through the points \( A(2,-3) \) and \( B(-1,2) \).

Let use \( P_0 \equiv A(2,-3) \) and \( \vec{u} = \overrightarrow{AB} = (-3,5) \). So:
\( \therefore L : \vec{r} = (2,-3) + t(-3,5), \quad t \in \mathbb{R} \)

Let use \( P_0 \equiv B(-1,2) \) and \( \vec{u} = 2\overrightarrow{BA} = 2(3,-5) = (6,-10) \). So:
\( \therefore L : \vec{r} = (-1,2) + s(6,-10), \quad s \in \mathbb{R} \)

### Ex 3

Convert each vector equation into the parametric equations.

- a) \( \vec{r} = (1,-3) + t(-2,5), \quad t \in \mathbb{R} \)
  \( (x,y) = (1,-3) + t(-2,5), \quad t \in \mathbb{R} \)
  \( \begin{cases} x = 1 - 2t \\ y = -3 + 5t \end{cases}, \quad t \in \mathbb{R} \)

- b) \( \vec{r} = (-2,0) + s(0,-3), \quad s \in \mathbb{R} \)
  \( (x,y) = (-2,0) + s(0,-3), \quad s \in \mathbb{R} \)
  \( \begin{cases} x = -2 \\ y = -3s \end{cases}, \quad s \in \mathbb{R} \)
Ex 4. Convert the parametric equations of each line into a vector equation.

a) \[
\begin{align*}
\begin{cases}
x = -2 + 3t \\
y = 5 - t
\end{cases}
\quad t \in \mathbb{R}
\end{align*}
\]

\[P_0 (2,5) \text{ and } \vec{u} = (3,-1)\]

\[\vec{r} = (-2,5) + t(3,-1), \quad t \in \mathbb{R}\]

b) \[
\begin{align*}
\begin{cases}
x = 1 - 3s \\
y = 2
\end{cases}
\quad s \in \mathbb{R}
\end{align*}
\]

\[P_0 (1,2) \text{ and } \vec{u} = (-3,0)\]

\[\vec{r} = (1,2) + s(-3,0), \quad s \in \mathbb{R}\]

C Parallel Lines

Two lines \(L_1\) and \(L_2\) with direction vectors \(\vec{u}_1\) and \(\vec{u}_2\) are parallel \((L_1 \parallel L_2)\) if:

\[\vec{u}_1 \parallel \vec{u}_2\]

or, there exists \(k \in \mathbb{R}\) such that:

\[\vec{u}_2 = k\vec{u}_1\]

or scalar components are proportional:

\[\frac{u_{2x}}{u_{1x}} = \frac{u_{2y}}{u_{1y}}\]

D Perpendicular Lines

Two lines \(L_1\) and \(L_2\) with direction vectors \(\vec{u}_1\) and \(\vec{u}_2\) are perpendicular \((L_1 \perp L_2)\) if:

\[\vec{u}_2 \perp \vec{u}_1\]

or:

\[\vec{u}_1 \cdot \vec{u}_2 = 0\]

or:

\[u_{1x}u_{2x} + u_{1y}u_{2y} = 0\]

E 2D Perpendicular Vectors

Given a 2D vector \(\vec{u} = (a,b)\), two 2D vectors perpendicular to \(\vec{u}\) are \(\vec{v} = (-b,a)\) and \(\vec{w} = (b,-a)\).

Indeed:

\[\vec{u} \cdot \vec{v} = a(-b) + b(a) = 0 \Rightarrow \vec{u} \perp \vec{v}\]

F Special Lines

A line parallel to the x-axis has a direction vector in the form \(\vec{u} = (u_x,0)\), \(u_x \neq 0\).

A line parallel to the y-axis has a direction vector in the form \(\vec{u} = (0,u_y)\), \(u_y \neq 0\).

---

Ex 5. Find the points of intersection between the line given by \(L_1 : \vec{r} = (1,2) + t(-1,1), \quad t \in \mathbb{R}\) and the coordinate axes.

\[L_1 : \begin{cases} x = 1 - t \\ y = 2 + t \end{cases} \quad t \in \mathbb{R}\]

Let \(A = L \cap x\)-axis (the intersection with the x-axis). Then

\[y_A = 0 \quad \Rightarrow \quad \begin{cases} x = 1 - t \\ 0 = 2 + t \end{cases} \Rightarrow t = -2 \Rightarrow x_A = 1 - (-2) = 3 \Rightarrow A(3,0)\]

Let \(B = L \cap y\)-axis (the intersection with the y-axis). Then

\[x_B = 0 \quad \Rightarrow \quad \begin{cases} 0 = 1 - t \\ y = 2 + t \end{cases} \Rightarrow y_B = 2 + 1 = 3 \Rightarrow B(0,3)\]

---

Ex 6. Consider the line \(L_1 : \vec{r} = (1,-3) + t(-1,2), \quad t \in \mathbb{R}\). Find the vector equation of a line \(L_2\), parallel to \(L_1\) that passes through the point \(M(-1,-13)\).

Because \(L_1 \parallel L_2\), a valid direction vector for \(L_2\) is \(\vec{u}_1 = (-1,2)\).

So, a possible vector equation for \(L_2\) is:

\[L_2 : \vec{r} = (-1,-13) + s(-1,2), \quad s \in \mathbb{R}\]

---

Ex 7. Show that \(L_1 \perp L_2\) where:

\[L_1 : \vec{r} = (-2,3) + t(2,-1), \quad t \in \mathbb{R}\]

\[L_2 : \begin{cases} x = -2s \\ y = 5 - 4s \end{cases} \quad s \in \mathbb{R}\]

\[\vec{u}_1 = (2,-1)\]

\[\vec{u}_2 = (-2,-4)\]

\[\vec{u}_1 \cdot \vec{u}_2 = 2(-2) + (-1)(-4) = -4 + 4 = 0 \Rightarrow L_1 \perp L_2\]

---

Ex 8. Consider the line \(L_1 : \vec{r} = (0,2) + t(2,-3), \quad t \in \mathbb{R}\). Find the vector equation of a line \(L_2\), perpendicular to \(L_1\) that passes through the point \(N(-3,0)\).

\[\vec{u}_1 = (2,-3); \quad \vec{u}_2 \perp \vec{u}_1 \Rightarrow \vec{u}_2 = (3,2)\]

\[L_2 : \vec{r} = (-3,0) + s(3,2), \quad s \in \mathbb{R}\]