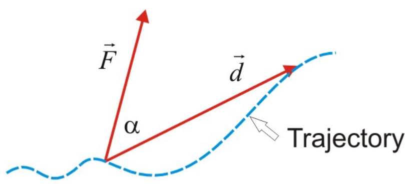
7.7 Applications of the Dot and Cross Product

A Work

The work W done by a *constant force* $\vec{F}$ acting on an object during a *displacement* $\vec{d}$ is given by:

$$W = \vec{F} \cdot \vec{d} = F\ d\ \cos\alpha$$



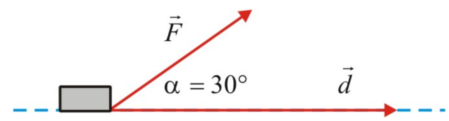
where

$$\alpha = \angle(\vec{F}, \vec{d})$$

$$[\vec{F}]_{SI} = N\ (Newton) \quad [\vec{d}]_{SI} = m\ (meter)$$

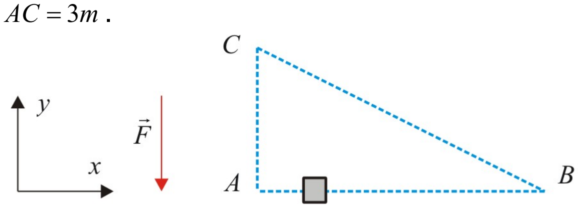
$$[W]_{SI} = J\ (Joule)$$

Ex 1. A box is pulled a horizontal distance of $100m$ by a force of $500N$ applied at an angle of $30°$ to the horizontal line. Calculate the amount of work done.



$$W = F\ d\ \cos\alpha = (500N)(100m)\cos 30° = 25000\sqrt{3}J \cong 43.3kJ$$

Ex 2. An object with a weight of $50N$ is moved, in vertical plane, along to the path $ABCA$ as presented in the next figure where $AB = 6m$ and $AC = 3m$.



Find the work done by the force of gravity:

a) from A to B

$\vec{F} = (0, -50)N$

$\vec{d} = \overrightarrow{AB} = (6,0)m$

$W = \vec{F} \cdot \vec{d} = (0)(6) + (-50)(0) = 0J$

b) from B to C

$\vec{F} = (0, -50)N$

$\vec{d} = \overrightarrow{BC} = (-6,3)m$

$W = \vec{F} \cdot \vec{d} = (0)(-6) + (-50)(3) = -150J$

c) from C to A

$\vec{F} = (0, -50)N$

$\vec{d} = \overrightarrow{CA} = (0,-3)m$

$W = \vec{F} \cdot \vec{d} = (0)(0) + (-50)(-3) = 150J$

d) along to the path $ABCA$

$\vec{F} = (0, -50)N$

$\vec{d} = \overrightarrow{AA} = \vec{0}$

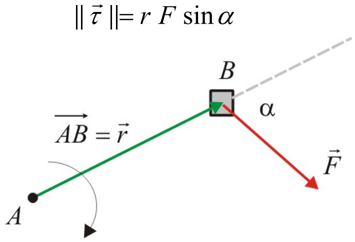
$W = \vec{F} \cdot \vec{d} = \vec{F} \cdot \vec{0} = 0J$

B Torque

The torque (rotational or turning effect) about the point A, created by a force $\vec{F}$ acting on an object located at the point B is given by:

$$\vec{\tau} = \overrightarrow{AB} \times \vec{F} = \vec{r} \times \vec{F}$$

$$\|\vec{\tau}\| = r\ F\ \sin\alpha$$



where:

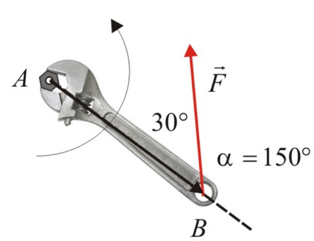
$$\alpha = \angle(\vec{F}, \vec{r})$$

$$\vec{r} = \overrightarrow{AB}\ (meter)$$

$$[\vec{F}]_{SI} = N\ (Newtom)$$

$$[\vec{\tau}]_{SI} = Nm$$

Ex 3. A wrench $30cm$ long is used to loose a bolt by applying a force of $20N$ (see the figure below). Find the magnitude of the torque.



$r = AB = 30cm = 0.3m$

$F = 20N$

$\alpha = \angle(\vec{F}, \overrightarrow{AB}) = 180° - 30° = 150°$

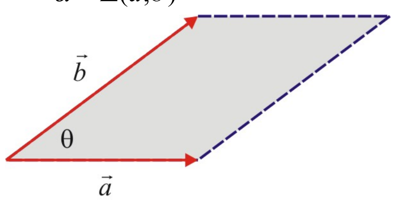
$\|\vec{\tau}\| = r\ F\ \sin\alpha = (0.3)(20)\sin 150° = 3Nm$

$\therefore \|\vec{\tau}\| = 3Nm$

C Parallelogram Area
The *area* of a *parallelogram* defined by the vectors $\vec a$ and $\vec b$ is determined by the formula:

$$A=\|\vec a\times\vec b\|=\|\vec a\|\|\vec b\|\sin\alpha$$

$$\alpha=\angle(\vec a,\vec b)$$



Ex 4. Find the area of the parallelogram defined by the vectors $\vec a=(1,-1,0)$ and $\vec b=(0,1,2)$.

$$\vec a\times\vec b=\begin{array}{ccccc}\vec i&\vec j&\vec k&\vec i&\vec j\\1&-1&0&1&-1\\0&1&2&0&1\end{array}=\vec i(-2-0)+\vec j(0-2)+\vec k(1-0)$$

$$=(-2,-2,1)$$

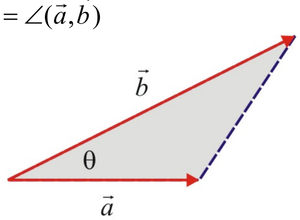
$$A=\|\vec a\times\vec b\|=\sqrt{(-2)^2+(-2)^2+1^2}=\sqrt9=3$$

$$\therefore A=3$$

D Triangle Area
The *area* of a *triangle* defined by the vectors $\vec a$ and $\vec b$ is given by:

$$A=\frac12\|\vec a\times\vec b\|=\frac12\|\vec a\|\|\vec b\|\sin\alpha$$

$$\alpha=\angle(\vec a,\vec b)$$



Ex 5. Find the area of the triangle ΔABC where $A(0,1,2)$, $B(-1,0,2)$, and $C(1,-2,0)$.

$$\overrightarrow{AB}=(-1,0,2)-(0,1,2)=(-1,-1,0)$$

$$\overrightarrow{AC}=(1,-2,0)-(0,1,2)=(1,-3,-2)$$

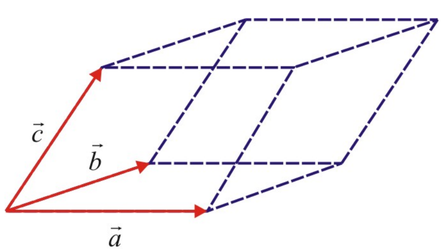
$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{array}{ccccc}\vec i&\vec j&\vec k&\vec i&\vec j\\-1&-1&0&-1&-1\\1&-3&-2&1&-3\end{array}=\vec i(2-0)+\vec j(0-2)+\vec k(3+1)$$

$$=(2,-2,4)$$

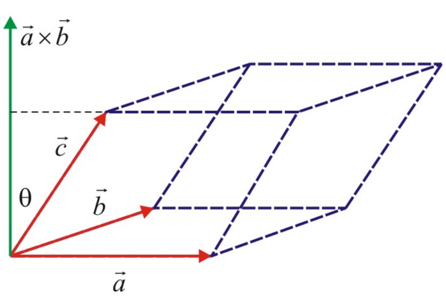
$$A=\frac12\|\vec a\times\vec b\|=\frac12\sqrt{2^2+(-2)^2+4^2}=\frac12\sqrt{24}=\sqrt6$$

E Parallelepiped Volume
The *volume* of a *parallelepiped* defined by the vectors $\vec a$, $\vec b$, and $\vec c$ is given by:

$$V=|\vec c\cdot(\vec a\times\vec b)|=|\vec a\cdot(\vec b\times\vec c)|=|\vec b\cdot(\vec c\times\vec a)|$$



Proof:



$$V=A_{base}\times h=\|\vec a\times\vec b\||SProj(\vec c\ onto\ \vec a\times\vec b)|$$

$$=\|\vec a\times\vec b\|\left|\frac{\vec c\cdot(\vec a\times\vec b)}{\|\vec a\times\vec b\|}\right|=|\vec c\cdot(\vec a\times\vec b)|$$

Ex 6. Find the volume of the parallelepiped defined by the vectors $\vec a=(0,1,-3)$, $\vec b=(1,2,3)$ and $\vec c=(-1,0,1)$.

$$\vec b\times\vec c=\begin{array}{ccccc}\vec i&\vec j&\vec k&\vec i&\vec j\\1&2&3&1&2\\-1&0&1&-1&0\end{array}=\vec i(2-0)+\vec j(-3-1)+\vec k(0+2)$$

$$=(2,-4,2)$$

$$\vec a\cdot(\vec b\times\vec c)=(0)(2)+(1)(-4)+(-3)(2)=-10$$

$$V=|\vec a\cdot(\vec b\times\vec c)|=|-10|=10$$

Ex 7. Find an unit vector perpendicular to both $\vec a=(0,1,1)$ and $\vec b=(1,1,0)$.

The vector $\vec u=\frac{\vec a\times\vec b}{\|\vec a\times\vec b\|}$ is an unit vector, perpendicular to both $\vec a$ and $\vec b$.

$$\vec a\times\vec b=\begin{array}{ccccc}\vec i&\vec j&\vec k&\vec i&\vec j\\0&1&1&0&1\\1&1&0&1&1\end{array}=\vec i(0-1)+\vec j(1-0)+\vec k(0-1)$$

$$=(-1,1,-1)$$

$$\therefore\vec u=\frac{\vec a\times\vec b}{\|\vec a\times\vec b\|}=\frac{(-1,1,-1)}{\sqrt3}=\left(\frac{-1}{\sqrt3},\frac{1}{\sqrt3},\frac{-1}{\sqrt3}\right)$$

Reading: Nelson Textbook, Pages 409-414
Homework: Nelson Textbook: Page 414 #3, 5a, 8, 10