7.6 Cross Product

A Right Hand System
The Right Hand System is based on the position of first three fingers of the right hand as illustrated on the following figure:

B Cork-Screw Rule
The cork-screw rule describes a right hand system based on the cork-screw property:

If you rotate the x-axis towards the y-axis using the shortest path, the screw goes in the positive direction of the z-axis.

C Cross Product
The cross product between two vectors \( \vec{a} \) and \( \vec{b} \) is a vector quantity denoted by \( \vec{a} \times \vec{b} \) having the following properties:

a) \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \alpha \) where \( \alpha = \angle(\vec{a}, \vec{b}) \)

b) \( \vec{a} \times \vec{b} \) is perpendicular to both \( \vec{a} \) and \( \vec{b} \) (is perpendicular to the plane determined by \( \vec{a} \) and \( \vec{b} \))

c) the vectors \( \vec{a} \), \( \vec{b} \), and \( \vec{a} \times \vec{b} \) form a right-handed system

D Specific Cases
1. If \( \vec{a} \parallel \vec{b} \) (\( \alpha = 0 \) or \( \alpha = \pi = 180^\circ \)), then \( \vec{a} \times \vec{b} = \vec{0} \).
2. If \( \vec{a} \perp \vec{b} \) (\( \alpha = \pi / 2 = 90^\circ \)), then \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| = \text{maximum} \)
3. If \( \vec{a} \equiv \vec{b} \) then \( \vec{a} \times \vec{a} = \vec{0} \).

Ex 1. The magnitudes of two vectors \( \vec{a} \) and \( \vec{b} \) are \( ||\vec{a}|| = 2 \) and \( ||\vec{b}|| = 3 \) respectively, and the angle between them is \( \alpha = 60^\circ \). Find the magnitude of the cross product of these vectors.

\[ ||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \alpha = 2 \times 3 \times \sin 60^\circ = 3 \sqrt{3} \]

E Cross Product of Unit Vectors
The cross product of the standard unit vectors is given by:

\[
\begin{align*}
\hat{i} \times \hat{i} &= \hat{0} \\
\hat{i} \times \hat{j} &= \hat{k} \\
\hat{i} \times \hat{k} &= \hat{j} \\
\hat{j} \times \hat{i} &= \hat{k} \\
\hat{j} \times \hat{j} &= \hat{0} \\
\hat{j} \times \hat{k} &= \hat{i} \\
\hat{k} \times \hat{i} &= \hat{j} \\
\hat{k} \times \hat{j} &= \hat{i} \\
\hat{k} \times \hat{k} &= \hat{0}
\end{align*}
\]

D Cross Product of two Algebraic Vectors
The cross product of two algebraic vectors

\[
\vec{a} = (a_x, a_y, a_z) = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}
\]

and

\[
\vec{b} = (b_x, b_y, b_z) = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}
\]

is given by:

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_x & a_y & a_z \\
b_x & b_y & b_z
\end{vmatrix}
\]

\[
= \begin{vmatrix}
a_y & a_z \\
b_y & b_z
\end{vmatrix} \hat{i} - \begin{vmatrix}
a_x & a_z \\
b_x & b_z
\end{vmatrix} \hat{j} + \begin{vmatrix}
a_x & a_y \\
b_x & b_y
\end{vmatrix} \hat{k}
\]

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
a_x & b_x \\
a_y & b_y
\end{vmatrix} \hat{i} - \begin{vmatrix}
a_x & a_z \\
b_x & b_z
\end{vmatrix} \hat{j} + \begin{vmatrix}
a_y & a_z \\
b_y & b_z
\end{vmatrix} \hat{k}
\]

\[
= \begin{vmatrix}
a_x & a_y \\
b_x & b_y
\end{vmatrix} \hat{i} - \begin{vmatrix}
a_x & a_z \\
b_x & b_z
\end{vmatrix} \hat{j} + \begin{vmatrix}
a_y & a_z \\
b_y & b_z
\end{vmatrix} \hat{k}
\]
Ex 2. For each case, find the cross product of the vectors \( \vec{a} \) and \( \vec{b} \).

a) \( \vec{a} = (1, -2, 0), \quad \vec{b} = (0, -1, 2) \)

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & -2 & 0 \\
0 & -1 & 2 \\
\end{vmatrix} = \mathbf{i}(-4-0) + \mathbf{j}(0-0) + \mathbf{k}(1-0) = \vec{a} \times \vec{b} = (0, 2, -1)
\]

b) \( \vec{a} = -\vec{i} + 2\vec{j} \), \( \vec{b} = \vec{i} - 2\vec{j} - \vec{k} \)

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 2 & 0 \\
1 & -2 & -1 \\
\end{vmatrix} = \mathbf{i}(-2-0) + \mathbf{j}(1-0) + \mathbf{k}(1-2) = \vec{a} \times \vec{b} = (0, 1, 1)
\]

\[
\therefore \vec{a} \times \vec{b} = (0, 2, -1)
\]

E Properties of Cross Product
The following properties apply for the cross product:
1. \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \) (anti-commutative property)
2. \( \lambda(\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) \)
3. \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \) (distributive property)
4. \( \vec{a} \times \vec{b} = 0 \iff \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \parallel \vec{b} \)
5. \( \vec{a} \times \vec{0} = \vec{0} \)
6. \( \vec{a} \times \vec{a} = \vec{0} \)

Note: The dot and cross products have a higher priority in comparison to addition and subtraction operations.

d) \( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c} \) (triple cross product)

| \( [\vec{a} \times (\vec{b} \times \vec{c})]_x \) | \( = a_x (\vec{b} \times \vec{c})_x - a_x (\vec{b} \times \vec{c})_y \\
| \( a_x (b_y c_z - b_z c_y) - a_x (b_z c_y - b_y c_z) \) | \( = (c_y a_z + c_z a_y) b_x - (b_y a_z + b_z a_y) c_x + a_x c_y b_z - a_x c_z b_y \) |

\[
= (\vec{c} \cdot \vec{a})b_x - (\vec{b} \cdot \vec{a})c_x = RS
\]

Ex 4. Find an unit vector perpendicular to both \( \vec{a} = (0,1,1) \) and \( \vec{b} = (1,0,0) \).

The vector \( \vec{u} = \frac{\vec{a} \times \vec{b}}{||\vec{a} \times \vec{b}||} \) is an unit vector perpendicular to both \( \vec{a} \) and \( \vec{b} \).

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{vmatrix} = \vec{0} - \vec{i} = \vec{a} \times \vec{b} = \vec{0} - \vec{i}
\]

\[
\therefore \vec{u} = \frac{(-1, 1, -1)}{\sqrt{3}} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)
\]

Reading: Nelson Textbook, Pages 401-407
Homework: Nelson Textbook: Page 407 #3, 4ab, 5, 8a, 11, 13