7.4 Dot Product of Algebraic Vectors

A Dot Product for Standard Unit Vectors
The *dot product* of the *standard unit vectors* is given by:

$\vec{i}\cdot\vec{i}=1$ $\quad \vec{j}\cdot\vec{j}=1$ $\quad \vec{k}\cdot\vec{k}=1$

$\vec{i}\cdot\vec{j}=0$ $\quad \vec{j}\cdot\vec{k}=0$ $\quad \vec{k}\cdot\vec{i}=0$

Proof:

$\vec{i}\cdot\vec{i}=\|\vec{i}\|\|\vec{i}\|\cos 0^\circ=(1)(1)(1)=1$

$\vec{i}\cdot\vec{j}=\|\vec{i}\|\|\vec{j}\|\cos 90^\circ=(1)(1)(0)=0$

B Dot Product for two Algebraic Vectors
The *dot product* of two *algebraic* vectors
$\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$ and
$\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$ is given by:

$$\vec{a}\cdot\vec{b}=a_xb_x+a_yb_y+a_zb_z$$

Proof:

$\vec{a}\cdot\vec{b}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\cdot(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})$

$=(a_xb_x)(\vec{i}\cdot\vec{i})+(a_xb_y(\vec{i}\cdot\vec{j})+(a_xb_z)(\vec{i}\cdot\vec{k})+$

$+(a_yb_x)(\vec{j}\cdot\vec{i})+(a_yb_y(\vec{j}\cdot\vec{j})+(a_yb_z)(\vec{j}\cdot\vec{k})+$

$+(a_zb_x)(\vec{k}\cdot\vec{i})+(a_zb_y(\vec{k}\cdot\vec{j})+(a_zb_z)(\vec{k}\cdot\vec{k})$

$=a_xb_x+a_yb_y+a_zb_z$

Ex 1. For each case, find the dot product of the vectors $\vec{a}$ and $\vec{b}$.

a) $\vec{a}=(1,-2,0)$, $\vec{b}=(0,-1,2)$

$\vec{a}\cdot\vec{b}=(1)(0)+(-2)(-1)+(0)(2)=2$

b) $\vec{a}=-\vec{i}+2\vec{j}$, $\vec{b}=\vec{i}-2\vec{j}-\vec{k}$

$\vec{a}\cdot\vec{b}=(-1)(1)+(2)(-2)+(0)(-1)=-1-4=-5$

c) $\vec{a}=(-1,1,-1)$, $\vec{b}=-\vec{i}+2\vec{j}-2\vec{k}$

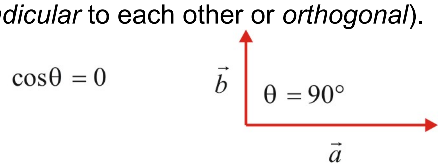
$\vec{a}\cdot\vec{b}=(-1)(-1)+(1)(2)+(-1)(-2)=1+2+2=5$

C Angle between two Vectors
The angle $\theta=\angle(\vec{a},\vec{b})$ between two vectors $\vec{a}$ and $\vec{b}$ (when positioned tail to tail) is given by:

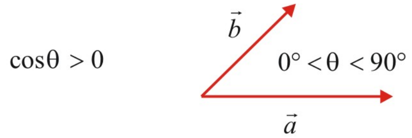
$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}$$

Notes:

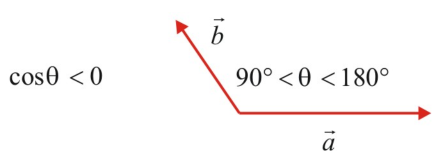
1. If $\cos\theta=1$ then $\vec{a}\uparrow\uparrow\vec{b}$ (vectors are *parallel* and have *same direction*).
2. If $\cos\theta=-1$ then $\vec{a}\uparrow\downarrow\vec{b}$ (vectors are *parallel* but have *opposite direction*).
3. If $\cos\theta=0$ then $\vec{a}\perp\vec{b}$ (vectors are *perpendicular* to each other or *orthogonal*).



4. If $\cos\theta>0$ then $0^\circ<\theta<90^\circ$ (θ is an *acute* angle).



5. If $\cos\theta<0$ then $90^\circ<\theta<180^\circ$ (θ is an *obtuse* angle).



Ex 2. For each case, find the angle between the vectors $\vec{a}$ and $\vec{b}$.

a) $\vec{a}=(1,-2,-1)$, $\vec{b}=(0,-1,2)$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(1)(0)+(-2)(-1)+(-1)(2)}{\sqrt{1^2+(-2)^2+(-1)^2}\sqrt{0^2+(-1)^2+2^2}}=0$$

$\therefore\theta=\cos^{-1}0=90^\circ \quad (\vec{a}\perp\vec{b})$

b) $\vec{a}=-\vec{i}-2\vec{k}$, $\vec{b}=-2\vec{j}+\vec{k}$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(-1)(0)+(0)(-2)+(-2)(1)}{\sqrt{(-1)^2+0^2+(-2)^2}\sqrt{0^2+(-2)^2+1^2}}=\frac{-2}{5}$$

$\therefore\theta=\cos^{-1}(-2/5)=113.58^\circ$

Ex 3. Find a non zero vector perpendicular to each of the vectors $\vec{a}=(1,5,-1)$ and $\vec{b}=(-3,1,2)$.

Let $\vec{v}=(x,y,z)$ be a vector perpendicular to both $\vec{a}$ and $\vec{b}$.
So:

$\vec{a}\cdot\vec{v}=0\Rightarrow x+5y-z=0 \quad (1)$

$\vec{b}\cdot\vec{v}=0\Rightarrow -3x+y+2z=0 \quad (2)$

$(1)\Rightarrow z=x+5y \quad (3)$

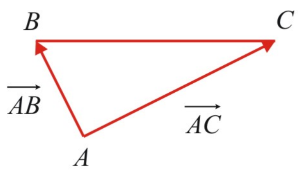
$(2)\Rightarrow -3x+y+2(x+5y)=0\Rightarrow 11y=x$

$(3)\Rightarrow z=11y+5y=16y$

$\vec{v}=(x,y,z)=(11y,y,16y)=y(11,1,16)$

$\therefore\vec{v}=y(11,1,16), \quad y\in R\setminus\{0\}$

Ex 4. A triangle is defined by three points $A(0,1,2)$, $B(1,0,2)$, and $C(-1,2,0)$. Find the angles $\angle A$ of this triangle.



$\overrightarrow{AB} = (1,0,2) - (0,1,2) = (1,-1,0)$

$\overrightarrow{AC} = (-1,2,0) - (0,1,2) = (-1,1,-2)$

$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\| \overrightarrow{AB} \| \| \overrightarrow{AC} \|} = \frac{-1-1+0}{\sqrt{2}\sqrt{6}} = \frac{-2}{\sqrt{12}} = \frac{-1}{\sqrt{3}}$

$\therefore \angle A = \cos^{-1}(-1/\sqrt{3}) \cong 125.26°$

Ex 5. Find the angles between the vector $\vec{a} = -2\vec{i} + \vec{j} + 3\vec{k}$ and the coordinate axes.

$\angle(\vec{a}, x-axis) = \angle(\vec{a}, \vec{i}) = \cos^{-1} \frac{\vec{a} \cdot \vec{i}}{\| \vec{a} \| \| \vec{i} \|} = \cos^{-1} \frac{a_x}{\| \vec{a} \|}$

$\therefore \angle(\vec{a}, x-axis) = \cos^{-1} \frac{-2}{\sqrt{14}} \cong 122.31°$

$\angle(\vec{a}, y-axis) = \angle(\vec{a}, \vec{j}) = \cos^{-1} \frac{\vec{a} \cdot \vec{j}}{\| \vec{a} \| \| \vec{j} \|} = \cos^{-1} \frac{a_y}{\| \vec{a} \|}$

$\therefore \angle(\vec{a}, y-axis) = \cos^{-1} \frac{1}{\sqrt{14}} \cong 74.5°$

$\angle(\vec{a}, z-axis) = \angle(\vec{a}, \vec{k}) = \cos^{-1} \frac{\vec{a} \cdot \vec{k}}{\| \vec{a} \| \| \vec{k} \|} = \cos^{-1} \frac{a_z}{\| \vec{a} \|}$

$\therefore \angle(\vec{a}, z-axis) = \cos^{-1} \frac{3}{\sqrt{14}} \cong 36.7°$

Ex 6. For what values of *k* are the vectors $\vec{a} = (k,-2,3)$ and $\vec{b} = (2,2k-6,6)$

a) perpendicular (orthogonal)?

$\vec{a} \cdot \vec{b} = 0 \Rightarrow 2k - 2(2k-6) + 18 = 0 \Rightarrow$

$2k - 4k + 12 + 18 = 0 \Rightarrow 30 = 2k \Rightarrow \therefore k = 15$

b) parallel (collinear)?

$\vec{a} = \lambda \vec{b} \Rightarrow (k,-2,3) = \lambda(2,2k-6,6) \Rightarrow$

$$\begin{cases} k = 2\lambda \\ -2 = \lambda(2k-6) \\ 3 = 6\lambda \Rightarrow \lambda = 0.5 \end{cases}$$

$k = 2(0.5) = 1$

$-2 = 0.5(2 \times 1 - 6) \Rightarrow -2 = -2 \ (true)$

$\therefore k = 1$

c) in opposite direction?

The vectors $\vec{a}$ and $\vec{b}$ are in opposite direction if there exists $\lambda < 0$ such that $\vec{a} = \lambda \vec{b}$. But, according to part b) if $\vec{a} = \lambda \vec{b}$ then $\lambda = 0.5 > 0$.

Therefore the vectors $\vec{a}$ and $\vec{b}$ cannot be in opposite direction for any real value of the parameter k.

Reading: Nelson Textbook, Pages 379-385

Homework: Nelson Textbook: Page #2c, 6d, 7a, 10, 13, 14, 18, 19

