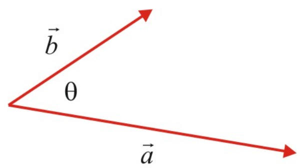
7.3 Dot Product of two Geometric Vectors

A Definition

The *Dot Product* of two *geometric* vectors $\vec{a}$ and $\vec{b}$ with an angle $\theta = \angle(\vec{a}, \vec{b})$ between them (when positioned tail to tail) is a *scalar* defined by:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

Note. By convention $0° \le \theta \le 180°$.



Ex 1. If $\|\vec{u}\| = 4$, $\|\vec{v}\| = 6$, and $\theta = \angle(\vec{u}, \vec{v}) = 120°$, find $\vec{u} \cdot \vec{v}$.

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta = (4)(6)\cos 120° = -12$

$\therefore \vec{u} \cdot \vec{v} = -12$

Ex 2. Find the angle between two unit vectors with a dot product equal to $1/\sqrt{2}$.

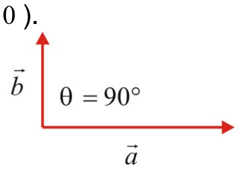
$\|\vec{u}\| = 1$, $\|\vec{v}\| = 1$, $\vec{u} \cdot \vec{v} = 1/\sqrt{2}$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta \Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{1/\sqrt{2}}{(1)(1)} = \frac{1}{\sqrt{2}}$

$\therefore \theta = \cos^{-1} \frac{1}{\sqrt{2}} = 45°$

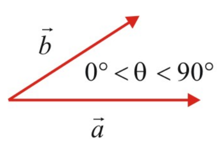
B Properties of Dot Product

1. $\vec{a} \cdot \vec{b}$ is a scalar (a real number).
2. If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$ (because $\theta = 90°$ and $\cos 90° = 0$).



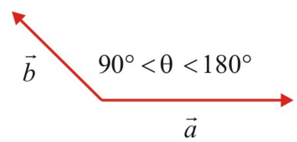
$\vec{a} \cdot \vec{b} = 0$

3. If $\vec{a} \cdot \vec{b} = 0$ then $\|\vec{a}\| = 0$ or $\|\vec{b}\| = 0$ or $\vec{a} \perp \vec{b}$.
4. If $0° < \theta < 90°$ then $\cos\theta > 0$ and $\vec{a} \cdot \vec{b} > 0$.



$\vec{a} \cdot \vec{b} > 0$

5. If $90° < \theta < 180°$ then $\cos\theta < 0$ and $\vec{a} \cdot \vec{b} < 0$.



$\vec{a} \cdot \vec{b} < 0$

6. If $\vec{a} \uparrow\uparrow \vec{b}$ then $\theta = 0°$, $\cos\theta = 1$, and $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$
7. If $\vec{a} \uparrow\downarrow \vec{b}$ then $\theta = 180°$, $\cos\theta = -1$, and $\vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\|$
8. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
9. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative property)
10. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (distributative property)
11. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$
12. $\vec{a} \cdot \vec{0} = 0$

Note. $\vec{0}$ is the zero vector and 0 is the number zero.

Ex 3. Use the dot product to prove each relation:

a) $\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b}$

$LS = \|\vec{a} + \vec{b}\|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b} = RS$

b) $\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a} \cdot \vec{b}$

$LS = \|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a} \cdot \vec{b} = RS$

c) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \|\vec{a}\|^2 - \|\vec{b}\|^2$

$LS = (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}$

$= \|\vec{a}\|^2 - \|\vec{b}\|^2 = RS$

Ex 4. If the vectors $2\vec{a} + \vec{b}$ and $\frac{\vec{a}}{2} - \vec{b}$ are perpendicular to each other and $2\|\vec{b}\| = 3\|\vec{a}\|$ find the angle $\theta = \angle(\vec{a}, \vec{b})$.

$2\|\vec{b}\| = 3\|\vec{a}\| \Rightarrow \frac{\|\vec{a}\|}{\|\vec{b}\|} = \frac{2}{3}$

$(2\vec{a} + \vec{b}) \cdot \left(\frac{\vec{a}}{2} - \vec{b}\right) = 0 \Rightarrow (2)\left(\frac{1}{2}\right)\vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \frac{1}{2}\vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 0$

$\|\vec{a}\|^2 - \|\vec{b}\|^2 = 2\vec{a} \cdot \vec{b} - \frac{1}{2}\vec{b} \cdot \vec{a} = \frac{3}{2}\vec{a} \cdot \vec{b} = \frac{3}{2}\|\vec{a}\| \|\vec{b}\| \cos\theta$

$\cos\theta = \frac{\|\vec{a}\|^2 - \|\vec{b}\|^2}{\frac{3}{2}\|\vec{a}\| \|\vec{b}\|} = \frac{\frac{\|\vec{a}\|^2}{\|\vec{b}\|^2} - 1}{\frac{3}{2}\frac{\|\vec{a}\|}{\|\vec{b}\|}} = \frac{\left(\frac{2}{3}\right)^2 - 1}{\frac{3}{2}\frac{2}{3}} = \frac{4}{9} - 1 = \frac{-5}{9}$

$\therefore \theta = \cos^{-1} \frac{-5}{9} = 123.75°$

Reading: Nelson Textbook, Pages 371-376

Homework: Nelson Textbook: Page 377 #1, 2, 5, 6b, 7a, 9a, 11, 15, 17

