6.7 Operations with Algebraic Vectors in R^3

A 3D Algebraic Vectors
A 3D Algebraic Vector may be written in components form as:

$$\vec{v} = (v_x, v_y, v_z)$$

or in terms of unit vectors as:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

and has a magnitude given by:

$$\|\vec{v}\| = \sqrt{v_x^{\ 2} + v_y^{\ 2} + v_z^{\ 2}}$$

Ex 1. Consider the vector $\vec{a} = -\vec{i} + 3\vec{j} - 2\vec{k}$.

a) Write the vector in components form.

$\vec{a} = -\vec{i} + 3\vec{j} - 2\vec{k} = (-1,3,-2)$

b) Find the magnitude of the vector $\vec{a}$.

$\|\vec{a}\| = \sqrt{a_x^{\ 2} + a_y^{\ 2} + a_z^{\ 2}} = \sqrt{(-1)^2 + 3^2 + (-2)^2} = \sqrt{14}$

B Addition of 3D Algebraic Vectors
The sum of two 3D algebraic vectors
$\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and
$\vec{b} = (b_x, b_y, b_z) = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is a 3D algebraic vector given by:

$$\vec{a} + \vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) + (b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$$
$$= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k}$$
$$\vec{a} + \vec{b} = (a_x, a_y, a_z) + (b_x, b_y, b_z)$$
$$= (a_x + b_x, a_y + b_y, a_z + b_z)$$

Ex 2. Find the sum of the vector $\vec{a} = -2\vec{i} + 5\vec{j} - \vec{k}$ and $\vec{b} = (2,0,-3)$.

$\vec{s} = \vec{a} + \vec{b} = (-2,5,-1) + (2,0,-3)$
$= (-2+2, 5+0, -1-3) = (0,5,-4) = 5\vec{j} - 4\vec{k}$

C Subtraction of 3D Algebraic Vectors
The difference of two 3D algebraic vectors
$\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and
$\vec{b} = (b_x, b_y, b_z) = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is a 3D algebraic vector given by:

$$\vec{a} - \vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) - (b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$$
$$= (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} + (a_z - b_z)\vec{k}$$
$$\vec{a} - \vec{b} = (a_x, a_y, a_z) - (b_x, b_y, b_z)$$
$$= (a_x - b_x, a_y - b_y, a_z - b_z)$$

Ex 3. Find the magnitude of the difference $\vec{a} - \vec{b}$ between the vector $\vec{a} = \vec{i} - \vec{j}$ and $\vec{b} = (1,2,-1)$.

$\vec{a} - \vec{b} = (1,-1,0) - (1,2,-1) = (1-1, -1-2, 0+1)$
$= (0,-3,1) = -3\vec{j} + \vec{k}$
$\|\vec{a} - \vec{b}\| = \sqrt{0^2 + (-3)^2 + 1^2} = \sqrt{10}$

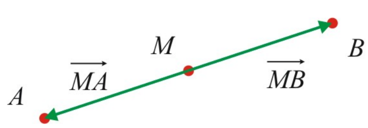
D Multiplication of 3D Algebraic Vector by a Scalar
The multiplication of a 3D algebraic vector
$\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ by a scalar λ is a 3D algebraic vector given by:

$$\lambda\vec{a} = \lambda(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) = (\lambda a_x)\vec{i} + (\lambda a_y)\vec{j} + (\lambda a_z)\vec{k}$$
$$\lambda\vec{a} = \lambda(a_x, a_y, a_z) = (\lambda a_x, \lambda a_y, \lambda a_z)$$

Ex 4. Given $\vec{a} = (1,-2,0)$, $\vec{b} = (0,-2,-3)$, and $\vec{c} = (-1,0,2)$, find the vector $\vec{d} = \vec{a} - 2\vec{b} + 3\vec{c}$.

$\vec{d} = \vec{a} - 2\vec{b} + 3\vec{c} = (1,-2,0) - 2(0,-2,-3) + 3(-1,0,2)$
$= (1-3, -2+4, 6+6) = (-2,2,12)$
$\therefore \vec{d} = (-2,2,12) = -2\vec{i} + 2\vec{j} + 12\vec{k}$

E Midpoint
The midpoint of the segment line AB is the point M such that $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$.



Ex 5. Prove that the midpoint of the segment line AB is given by:

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right)$$

$\overrightarrow{MA} + \overrightarrow{MB} = \vec{0} \Rightarrow \overrightarrow{OA} - \overrightarrow{OM} + \overrightarrow{OB} - \overrightarrow{OM} = \vec{0}$

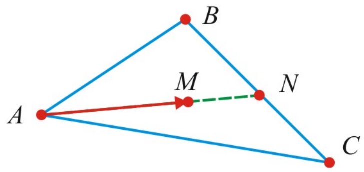
$\overrightarrow{OA} + \overrightarrow{OB} = 2\overrightarrow{OM} \Rightarrow \overrightarrow{OM} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$

F Centroid

The centroid of a system of points P_1, P_2, …, P_n is the point C defined by:

$$\overrightarrow{OC}=\frac{\overrightarrow{OP_1}+\overrightarrow{OP_2}+\ldots+\overrightarrow{OP_n}}{n}$$

Ex 6. Consider the triangle ΔABC where $A(-1,-4,1)$, $B(2,-3,0)$, and $C(-4,1,2)$.



a) Find the centroid M of the triangle.

$$\overrightarrow{OM}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}=\frac{(-1,-4,1)+(2,-3,0)+(-4,1,2)}{3}$$

$$=\frac{(-1+2-4,-4-3+1,1+0+2)}{3}=\frac{(-3,-6,3)}{3}=(-1,-2,1)$$

$\therefore M(-1,-2,1)$

b) Use the result at part a) to show that $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=\vec{0}$.

$\overrightarrow{MA}=(-1,-4,1)-(-1,-2,1)=(0,-2,0)$

$\overrightarrow{MB}=(2,-3,0)-(-1,-2,1)=(3,-1,-1)$

$\overrightarrow{MC}=(-4,1,2)-(-1,-2,1)=(-3,3,1)$

$\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=(0,-2,0)+(3,-1,-1)+(-3,3,1)=\vec{0}$

c) Find the midpoint N of the side BC.

$$\overrightarrow{ON}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}=\frac{(2,-3,0)+(-4,1,2)}{2}=\frac{(-2,-2,2)}{2}=(-1,-1,1)$$

$\therefore N(-1,-1,1)$

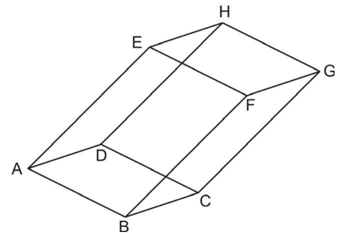
d) Show that $\overrightarrow{AN}=3\overrightarrow{MN}$.

$\overrightarrow{MN}=(-1,-1,1)-(-1,-2,1)=(0,1,0)$

$\overrightarrow{AN}=(-1,-1,1)-(-1,-4,1)=(0,3,0)$

$3\overrightarrow{MN}=3(0,1,0)=(0,3,0)=\overrightarrow{AN}$

Ex 7. The shape $ABCDEFGH$ is a parallelepiped. Given $A(0,1,3)$, $B(1,0,2)$, $C(1,2,0)$, and $E(4,4,4)$, find the coordinates of all the other vertices. See the figure below.



$\overrightarrow{OA}=(0,1,3)$, $\overrightarrow{OB}=(1,0,2)$, $\overrightarrow{OC}=(1,2,0)$, $\overrightarrow{OE}=(4,4,4)$

$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=\overrightarrow{OA}+\overrightarrow{BC}$

$=(0,1,3)+(1,2,0)-(1,0,2)=(0,3,1)$

$\therefore D(0,3,1)$

$\overrightarrow{OF}=\overrightarrow{OE}+\overrightarrow{EF}=\overrightarrow{OE}+\overrightarrow{AB}$

$=(4,4,4)+(1,0,2)-(0,1,3)=(5,3,3)$

$\therefore F(5,3,3)$

$\overrightarrow{OG}=\overrightarrow{OE}+\overrightarrow{EG}=\overrightarrow{OE}+\overrightarrow{AC}$

$=(4,4,4)+(1,2,0)-(0,1,3)=(5,5,1)$

$\therefore G(5,5,1)$

$\overrightarrow{OH}=\overrightarrow{OE}+\overrightarrow{EH}=\overrightarrow{OE}+\overrightarrow{BC}$

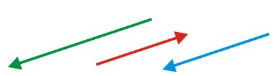
$=(4,4,4)+(0,2,-2)=(4,6,2)$

$\therefore H(4,6,2)$

G Parallelism

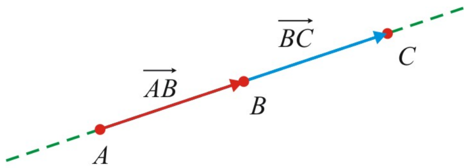
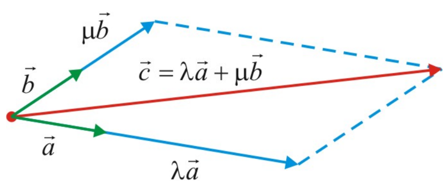
Two vectors $\vec{a}$ and $\vec{b}$ are parallel ($\vec{a}\parallel\vec{b}$) if there exists λ such that $\vec{a}=\lambda\vec{b}$.

Note that parallel vectors may have same direction or opposite direction:



Ex 8. Prove that the vectors $\vec{a}=(2,4,-6)$ and $b(-1,-2,3)$ are parallel.

$\vec{a}=(2,4,-6)=-2(-1,-2,3)=-2\vec{b}\Rightarrow\therefore\vec{a}\parallel\vec{b}$

H Co-linearity Three points A, B, and C are collinear if $\overrightarrow{AB} \parallel \overrightarrow{BC}$.  	Ex 9. Prove that the points $A(2,-1,0)$, $B(-1,0,2)$, and $C(0,1,2)$ are not collinear. $\overrightarrow{AB} = (-1,0,2) - (2,-1,0) = (-3,1,2)$ $\overrightarrow{BC} = (0,1,2) - (-1,0,2) = (1,1,0)$ $\overrightarrow{AB}$ is not parallel to $\overrightarrow{BC}$. Therefore the points A, B, and C are not collinear.
I Linear Dependency Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linear dependent if there exist λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$. Note. In order to be linear dependant the vectors must be coplanar (must belong to the same plan).  	Ex 10. Prove that the vectors $\vec{a} = (-1,2,-3)$, $\vec{b} = (2,0,-1)$, and $\vec{c} = (-7,6,-7)$ are linear dependant. Let try to find two scalars λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$. $(-7,6,-7) = \lambda(-1,2,-3) + \mu(2,0,-1)$ $\begin{cases} -7 = -\lambda + 2\mu & (1) \\ 6 = 2\lambda & (2) \\ -7 = -3\lambda - \mu & (3) \end{cases}$ $(2) \Rightarrow \lambda = 3 \quad (4)$ $(4) \Rightarrow (1): -7 = -3 + 2\mu \Rightarrow \mu = -2 \quad (5)$ $(4),(5) \Rightarrow (3): -7 = -3(3) - (-2) \quad (true)$ $\vec{c} = 3\vec{a} - 2\vec{b}$ Therefore, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linear dependent.

Reading: Nelson Textbook, Pages 327-332
Homework: Nelson Textbook: Page 332 #1, 3, 5b, 7c, 11, 12, 15