6.4 Properties of Vectors

A Properties of Vectors

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$
$$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$$
$$\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$$
$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$
$$\|k\vec{a}\|=|k|\,\|\vec{a}\|$$
$$k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$$
$$(kl)\vec{a}=k(l\vec{a})=l(k\vec{a})$$
$$(k+l)\vec{a}=k\vec{a}+l\vec{a}$$
$$1\vec{a}=\vec{a}$$
$$(-1)\vec{a}=-\vec{a}$$
$$0\vec{a}=\vec{0}$$
$$\|\vec{0}\|=0$$

Ex 1. If $\vec{a}=2\vec{i}-3\vec{j}$ and $\vec{b}=\vec{i}+\vec{j}$, find $\vec{i}$ and $\vec{j}$ in terms of $\vec{a}$ and $\vec{b}$.

$\vec{a}=2\vec{i}-3\vec{j}$ (1)
$\vec{b}=\vec{i}+\vec{j}$ (2)
From (2): $\vec{j}=\vec{b}-\vec{i}$ (3)
Let substitute (3) into (1):

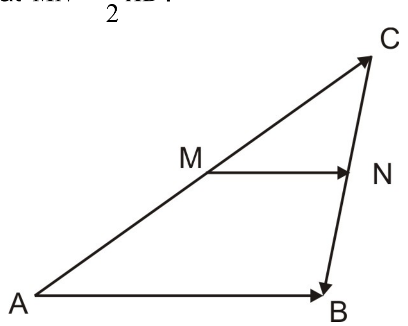
$\vec{a}=2\vec{i}-3(\vec{b}-\vec{i})\Rightarrow \vec{a}+3\vec{b}=5\vec{i}\Rightarrow \vec{i}=\dfrac{\vec{a}+3\vec{b}}{5}$ (4)

Let substitute (4) into (3):

$$\vec{j}=\vec{b}-\frac{\vec{a}+3\vec{b}}{5}=\frac{5\vec{b}-\vec{a}-3\vec{b}}{5}=\frac{2\vec{b}-\vec{a}}{5}$$

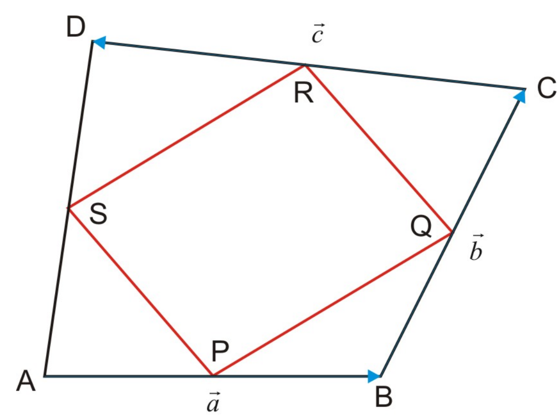
$$\therefore \vec{i}=\frac{\vec{a}+3\vec{b}}{5},\quad \vec{j}=\frac{-\vec{a}+2\vec{b}}{5}$$

Ex 2. Consider the triangle ΔABC. Let M be the midpoint of AC and N be the midpoint of BC.
Prove that $\overrightarrow{MN}=\frac{1}{2}\overrightarrow{AB}$.



$$\overrightarrow{AC}+\overrightarrow{CB}=\overrightarrow{AB}$$
$$\overrightarrow{MN}=\overrightarrow{MC}+\overrightarrow{CN}=\frac{1}{2}\overrightarrow{AC}+\frac{1}{2}\overrightarrow{CB}$$
$$=\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{CB})=\frac{1}{2}\overrightarrow{AB}$$
$$\therefore \overrightarrow{MN}=\frac{1}{2}\overrightarrow{AB}$$

Ex 3. Consider a polygon $ABCD$ and let P, Q, R, and S be the midpoints of AB, BC, CD, and DA respectively.
Prove that $PQRS$ is a parallelogram.

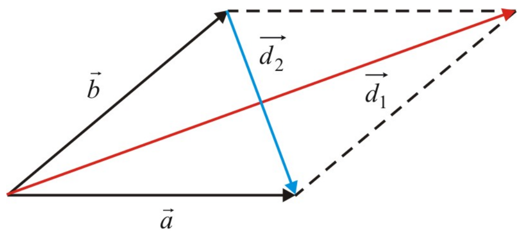


Let $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{BC}$, and $\vec{c}=\overrightarrow{CD}$. Then $\overrightarrow{AD}=\vec{a}+\vec{b}+\vec{c}$.

$$\overrightarrow{PQ}=\overrightarrow{PB}+\overrightarrow{BQ}=\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}=\frac{\vec{a}+\vec{b}}{2}$$
$$\overrightarrow{SR}=\overrightarrow{SD}+\overrightarrow{DR}=\frac{1}{2}\overrightarrow{AD}+\frac{1}{2}\overrightarrow{DC}=\frac{1}{2}\overrightarrow{AD}-\frac{1}{2}\overrightarrow{CD}$$
$$=\frac{1}{2}(\vec{a}+\vec{b}+\vec{c})-\frac{1}{2}\vec{c}=\frac{\vec{a}+\vec{b}}{2}$$
$$\overrightarrow{PQ}=\overrightarrow{SR}\Rightarrow PQ\parallel SR$$

Similarly, $PS\parallel QR$.
$\therefore$ $PQRS$ is a parallelogram.

Ex 4. Prove that diagonals of a rhombus (rhomboid) are perpendicular to each other.



For a rhomboid $\|\vec{a}\| = \|\vec{b}\|$.

$\overrightarrow{d_1} = \vec{a} + \vec{b}$

$\overrightarrow{d_2} = \vec{a} - \vec{b}$

$\overrightarrow{d_1} + \overrightarrow{d_2} = 2\vec{a}$

$\overrightarrow{d_1} - \overrightarrow{d_2} = 2\vec{b}$

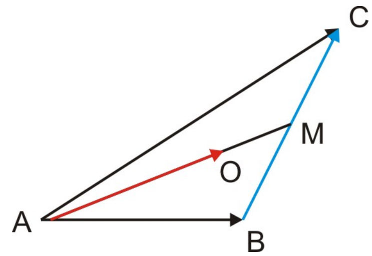
$\|\vec{a}\| = \|\vec{b}\| \Rightarrow \|\overrightarrow{d_1} + \overrightarrow{d_2}\| = \|\overrightarrow{d_1} - \overrightarrow{d_2}\|$

According to a previous proved fact, if the sum and the difference of two vectors have the same magnitude, then these two vectors are perpendicular.

$\therefore \overrightarrow{d_1} \perp \overrightarrow{d_2}$

Ex 5. Consider the triangle ΔABC and the point O defined by $\overrightarrow{AO} = \dfrac{\overrightarrow{AB} + \overrightarrow{AC}}{3}$. Let M be the midpoint of BC.

a) Prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{0}$.



$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + (\overrightarrow{OA} + \overrightarrow{AB}) + (\overrightarrow{OA} + \overrightarrow{AC})$

$= 3\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AC} = 3\overrightarrow{OA} + 3\overrightarrow{AO} = \vec{0}$

b) Prove that $\overrightarrow{AM} = \dfrac{3}{2}\overrightarrow{AO}$.

$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AB} + \dfrac{1}{2}\overrightarrow{BC}$

$= \overrightarrow{AB} + \dfrac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) = \overrightarrow{AB} + \dfrac{1}{2}(-\overrightarrow{AB} + \overrightarrow{AC})$

$= \dfrac{\overrightarrow{AB} + \overrightarrow{AC}}{2} = \dfrac{1}{2}(3\overrightarrow{AO}) = \dfrac{3}{2}\overrightarrow{AO}$

Reading: Nelson Textbook, Pages 302-306
Homework: Nelson Textbook: Page 306 #1, 6, 7, 8, 9;   Page 308 #3, 6, 7, 13, 15

