

6.2 Addition and Subtraction of Geometric Vectors

A Addition of two Vectors

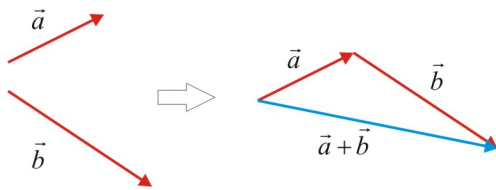
The *vector addition* \vec{s} of two vectors \vec{a} and \vec{b} is denoted by $\vec{a} + \vec{b}$ and is called the *sum* or *resultant* of the two vectors. So:

$$\vec{s} = \vec{a} + \vec{b}$$

B Triangle Rule (Tail to Tip Rule)

In order to find the sum (resultant) of two geometric vectors:

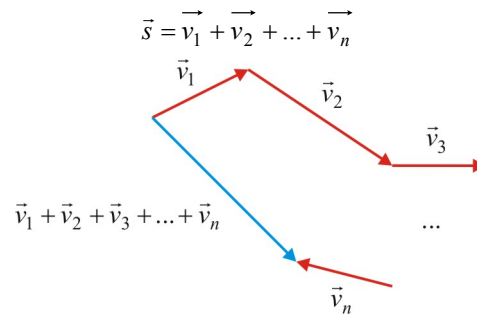
- Place the second vector with its tail on the tip (head) of the first vector.
- The sum (resultant) is a vector with the tail at the tail of the first vector and the head at the head of the second vector.



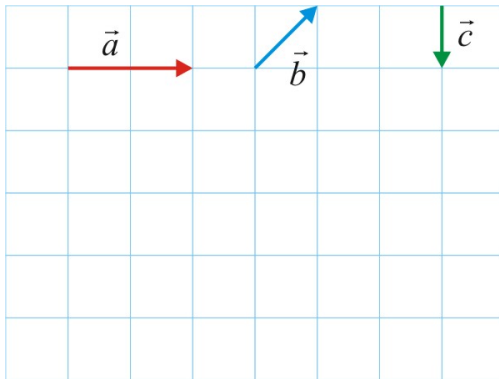
C Polygon Rule

In order to find the sum (resultant) of n geometric vectors:

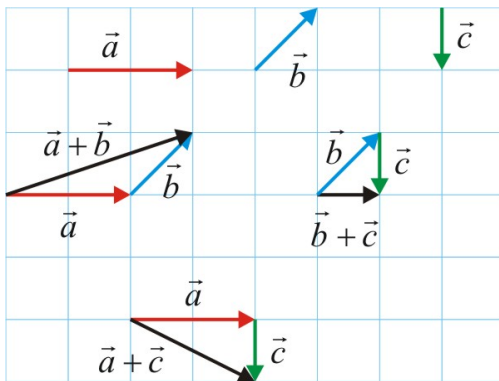
- Place the next vector with its tail on the tip (head) of the precedent vector.
- The sum (resultant) is a vector with the tail at the tail of the first vector and the head at the head of the last vector.



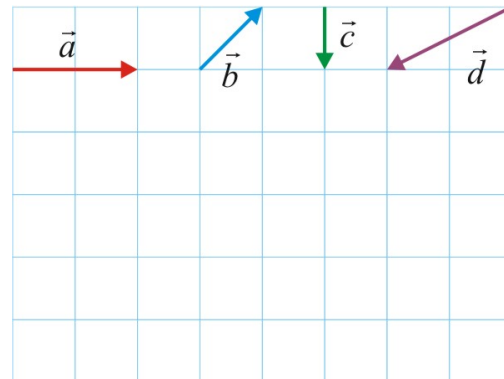
Ex 1. Use the following diagram and the triangle rule compute the required operations.



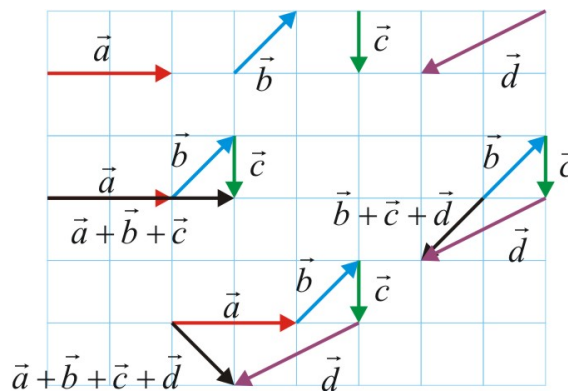
- a) $\vec{a} + \vec{b}$ b) $\vec{b} + \vec{c}$ c) $\vec{a} + \vec{c}$



Ex 2. Use the following diagram and the triangle rule compute the required operations.



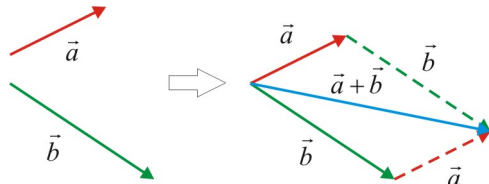
- a) $\vec{a} + \vec{b} + \vec{c}$ b) $\vec{b} + \vec{c} + \vec{d}$ c) $\vec{a} + \vec{b} + \vec{c} + \vec{d}$



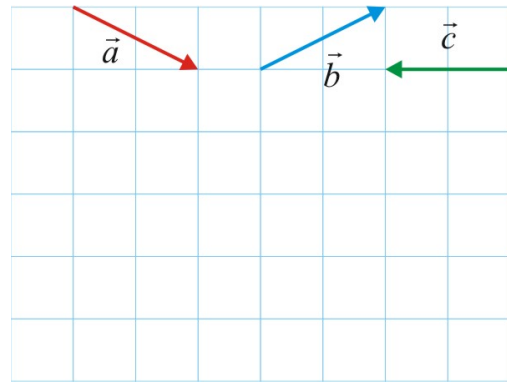
D Parallelogram Rule (Tail to Tail Rule)

To add two geometric vectors, the following rule can also be used:

- Position both vectors with their *tails* at the same point.
- Build a *parallelogram* using the vectors as two sides.
- The sum (resultant) is given by the *diagonal* of the parallelogram starting from the common tail point.



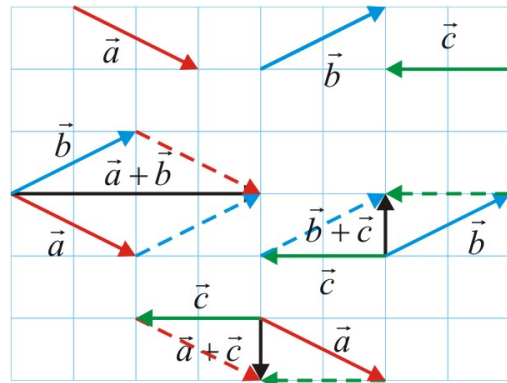
Ex 3. Use the parallelogram rule to compute the required operations:



a) $\vec{a} + \vec{b}$

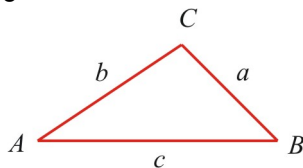
b) $\vec{b} + \vec{c}$

c) $\vec{a} + \vec{c}$



E Sine Law

For any triangle $\triangle ABC$



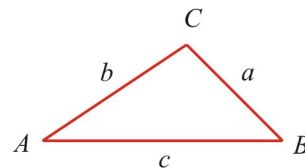
The following relation is true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This is called the *sine law*.

F Cosine Law

For any triangle $\triangle ABC$



The following relations are true:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

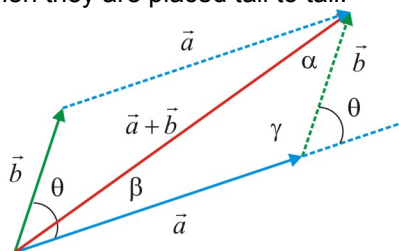
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2abc \cos C$$

Any of these is called the *cosine law*.

G Magnitude and Direction for Vector Sum

Let $\theta = \angle(\vec{a}, \vec{b})$ be the angle between the vectors \vec{a} and \vec{b} when they are placed tail to tail.



The magnitude of the vector sum is given by:

$$\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos \gamma \quad (1)$$

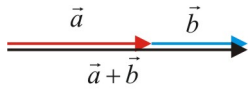
$$\gamma = 180^\circ - \theta$$

The direction of the vector sum $\vec{s} = \vec{a} + \vec{b}$ is defined by the angles α and β formed by the vector sum and the vectors \vec{b} and \vec{a} respectively:

$$\frac{\|\vec{a}\|}{\sin \alpha} = \frac{\|\vec{b}\|}{\sin \beta} = \frac{\|\vec{a} + \vec{b}\|}{\sin \gamma} \quad (2)$$

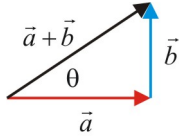
Ex 4. Analyze the following specific cases of addition of two vectors. Explain and make a diagram.

a) $\theta = 0$



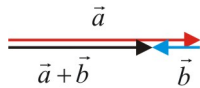
$$\|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$$

b) $\theta = \pi/2$



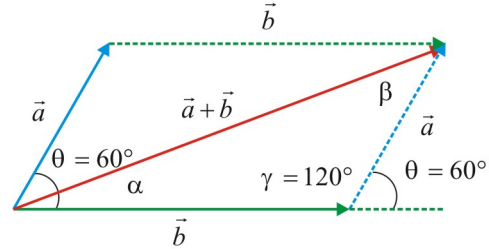
$$\|\vec{a} + \vec{b}\| = \sqrt{\|\vec{a}\|^2 + \|\vec{b}\|^2}, \quad \tan \theta = \frac{\|\vec{b}\|}{\|\vec{a}\|}$$

c) $\theta = \pi$



$$\|\vec{a} + \vec{b}\| = \|\|\vec{a}\| - \|\vec{b}\|\|$$

Ex 5. Given the magnitude of two vectors $\|\vec{a}\| = 4$ and $\|\vec{b}\| = 7$, and the angle between them when placed tail to tail as being $\theta = 60^\circ$, find the magnitude of the vector sum $\vec{s} = \vec{a} + \vec{b}$ and the direction (the angles between the vector sum and each vector). Draw a diagram.



$$\gamma = 180^\circ - \theta = 180^\circ - 60^\circ = 120^\circ$$

$$\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos \gamma$$

$$= 4^2 + 7^2 - 2(4)(7)\cos 120^\circ = 93$$

$$\|\vec{a} + \vec{b}\| = \sqrt{93} \approx 9.644$$

$$\frac{\|\vec{a}\|}{\sin \alpha} = \frac{\|\vec{a} + \vec{b}\|}{\sin \gamma} \Rightarrow \frac{4}{\sin \alpha} = \frac{\sqrt{93}}{\sin 120^\circ}$$

$$\alpha = \sin^{-1}\left(\frac{4\sin 60^\circ}{\sqrt{93}}\right) \approx 21.05^\circ$$

$$\beta = 180^\circ - 120^\circ - 21.05^\circ = 38.95^\circ$$

Ex 6. Given $\|\vec{a}\| = 4$, $\|\vec{b}\| = 6$, and $\|\vec{a} + \vec{b}\| = 8$, find the angle between the vectors \vec{a} and \vec{b} (when placed tail to tail) $\theta = \angle(\vec{a}, \vec{b})$.

$$\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos \gamma$$

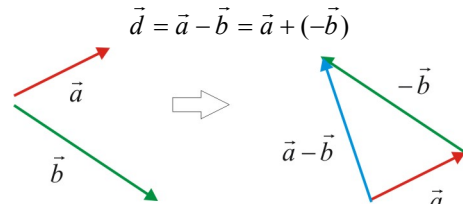
$$\cos \gamma = \frac{\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a} + \vec{b}\|^2}{2\|\vec{a}\|\|\vec{b}\|} = \frac{4^2 + 6^2 - 8^2}{2(4)(6)} = -0.25$$

$$\gamma = \cos^{-1}(-0.25) = 104.48^\circ$$

$$\theta = 180^\circ - \gamma = 180^\circ - 104.48^\circ = 75.52^\circ$$

H Vector Subtraction

The subtraction operation between two vectors $\vec{a} - \vec{b}$ can be understood as a vector addition between the first vector and the opposite of the second vector:

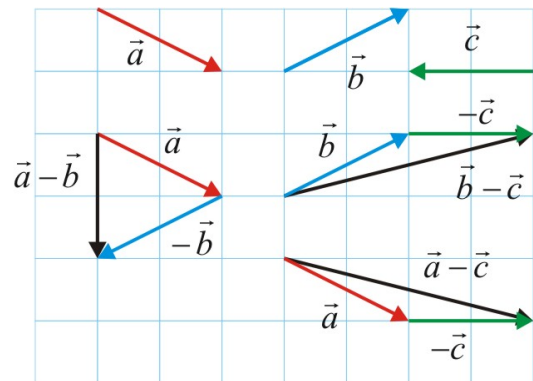
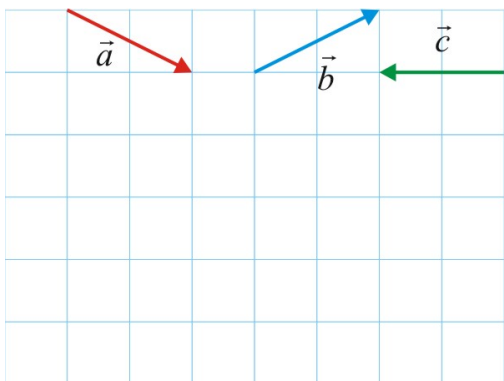


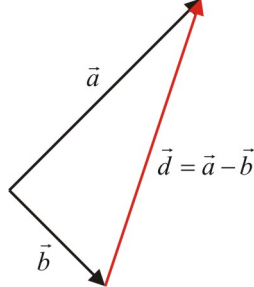
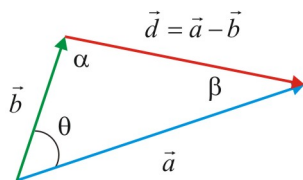
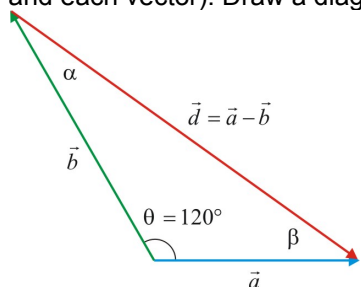
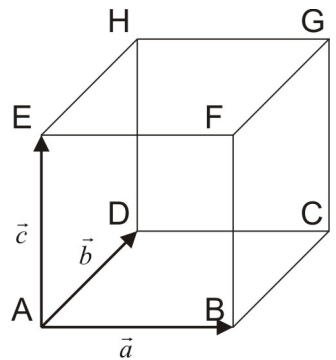
Ex 7. Compute the required operations.

a) $\vec{a} - \vec{b}$

b) $\vec{b} - \vec{c}$

c) $\vec{a} - \vec{c}$



<p>I Inverse Operation The vector subtraction operation is the inverse operation of the vector addition:</p> $\vec{d} = \vec{a} - \vec{b} \Leftrightarrow \vec{a} = \vec{b} + \vec{d}$	
<p>J Magnitude and Direction for Vector Difference Let \vec{a} and \vec{b} be two vectors and $\vec{d} = \vec{a} - \vec{b}$ be the vector difference. Let θ be the angle between the vectors \vec{a} and \vec{b} when they are placed tail to tail:</p>  <p>The magnitude of the <i>vector difference</i> is given by:</p> $\ \vec{a} - \vec{b}\ ^2 = \ \vec{a}\ ^2 + \ \vec{b}\ ^2 - 2\ \vec{a}\ \ \vec{b}\ \cos\theta \quad (3)$ <p>The direction of the vector difference $\vec{d} = \vec{a} - \vec{b}$ is defined by the angles α and β formed by the vector sum and the vectors \vec{b} and \vec{a} respectively:</p> $\frac{\ \vec{a}\ }{\sin\alpha} = \frac{\ \vec{b}\ }{\sin\beta} = \frac{\ \vec{a} - \vec{b}\ }{\sin\theta} \quad (4)$	<p>Ex 8. Given the magnitude of two vectors $\ \vec{a}\ = 10$ and $\ \vec{b}\ = 14$, and the angle between them when placed tail to tail as being $\theta = 120^\circ$, find the magnitude of the vector difference $\vec{d} = \vec{a} - \vec{b}$ and the direction (the angles between the vector sum and each vector). Draw a diagram.</p>  $\ \vec{a} - \vec{b}\ ^2 = \ \vec{a}\ ^2 + \ \vec{b}\ ^2 - 2\ \vec{a}\ \ \vec{b}\ \cos\theta$ $= 10^2 + 14^2 - 2(10)(14)\cos 120^\circ = 436$ $\ \vec{a} - \vec{b}\ = \sqrt{436} \approx 20.88$ $\frac{\ \vec{a}\ }{\sin\alpha} = \frac{\ \vec{a} - \vec{b}\ }{\sin\theta} \Rightarrow \frac{10}{\sin\alpha} = \frac{\sqrt{436}}{\sin 120^\circ}$ $\alpha = \sin^{-1} \frac{10 \sin 120^\circ}{\sqrt{436}} \approx 24.50^\circ$ $\beta = 180^\circ - 120^\circ - 24.50^\circ = 35.50^\circ$
<p>Ex 9. Given $\ \vec{a}\ = 10$, $\ \vec{b}\ = 15$, and $\ \vec{a} + \vec{b}\ = 20$, find $\ \vec{a} - \vec{b}\$.</p> $\ \vec{a} + \vec{b}\ ^2 = \ \vec{a}\ ^2 + \ \vec{b}\ ^2 - 2\ \vec{a}\ \ \vec{b}\ \cos\gamma$ $= \ \vec{a}\ ^2 + \ \vec{b}\ ^2 - 2\ \vec{a}\ \ \vec{b}\ \cos(180^\circ - \theta)$ $= \ \vec{a}\ ^2 + \ \vec{b}\ ^2 + 2\ \vec{a}\ \ \vec{b}\ \cos\theta$ $\ \vec{a} - \vec{b}\ ^2 = \ \vec{a}\ ^2 + \ \vec{b}\ ^2 - 2\ \vec{a}\ \ \vec{b}\ \cos\theta$ $\ \vec{a} + \vec{b}\ ^2 + \ \vec{a} - \vec{b}\ ^2 = 2\ \vec{a}\ ^2 + 2\ \vec{b}\ ^2$ $\ \vec{a} - \vec{b}\ ^2 = 2\ \vec{a}\ ^2 + 2\ \vec{b}\ ^2 - \ \vec{a} + \vec{b}\ ^2$ $= 2(10^2) + 2(15^2) - 20^2 = 200 + 450 - 400 = 250$ $\ \vec{a} - \vec{b}\ = \sqrt{250} = 5\sqrt{10}$	<p>Ex 10. A cube is constructed from three vectors \vec{a}, \vec{b}, and \vec{c}, as shown in the left figure. Express in terms of \vec{a}, \vec{b}, and \vec{c}:</p> <p>a) \vec{AF} $\vec{AF} = \vec{a} + \vec{c}$</p> <p>b) \vec{CF} $\vec{CF} = -\vec{b} + \vec{c}$</p> <p>c) \vec{AG} $\vec{AG} = \vec{a} + \vec{b} + \vec{c}$</p> <p>d) \vec{BH} $\vec{BH} = -\vec{a} + \vec{b} + \vec{c}$</p> 

Reading: Nelson Textbook, Pages 282-289

Homework: Nelson Textbook: Page 290 #2, 5, 7, 12, 13, 14, 16