5A Derivative of Logarithmic Function

A Review of Logarithmic Function

$$y = b^x \quad \Leftrightarrow \quad x = \log_b y$$

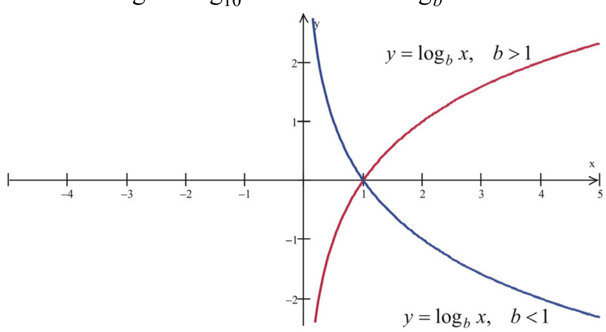
$$y = f(x) = \log_b x, \quad b > 0, b \neq 1, x > 0$$

$$\log_b(xy) = \log_b x + \log_b x \qquad \log_b x^n = n \log_b x$$

$$\log_b \frac{x}{y} = \log_b x - \log_b x \qquad \log_b x = \frac{\log_a x}{\log_b a}$$

$$\ln x = \log_e x \qquad \log_b 1 = 0$$

$$\log x = \log_{10} x \qquad \log_b b = 1$$



Ex 1. Use the graph of the logarithmic function to evaluate each limit.

a) $\lim_{x \to 0^+} \ln x$

$\lim_{x \to 0^+} \ln x = -\infty$

b) $\lim_{x \to 0^+} \log_{0.5} x$

$\lim_{x \to 0^+} \log_{0.5} x = \infty$

c) $\lim_{x \to \infty} \log x$

$\lim_{x \to \infty} \log x = \infty$

d) $\lim_{x \to \infty} \log_{0.1} x$

$\lim_{x \to \infty} \log_{0.1} x = -\infty$

B Derivative of $\ln x$

$$(\ln x)' = \frac{1}{x}$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad (1)$$

Proof:

$$y = \ln x \quad \Rightarrow \quad x = e^y \quad \Rightarrow \quad (x)' = (e^y)' \quad \Rightarrow$$

$$1 = e^y y' \quad \Rightarrow \quad y' = \frac{1}{e^y} \quad \Rightarrow \quad y' = \frac{1}{x} \quad \Rightarrow \quad \therefore (\ln x)' = \frac{1}{x}$$

Ex 2. Differentiate and simplify.

a) $x^2 \ln x$

$$(x^2 \ln x)' = (x^2)' \ln x + x^2 (\ln x)' = 2x \ln x + x^2 \frac{1}{x}$$

$$= 2x \ln x + x = x(2 \ln x + 1)$$

b) $\frac{\ln x}{x}$

$$\left(\frac{\ln x}{x}\right)' = \frac{(\ln x)' x - (\ln x)(x)'}{x^2} = \frac{\frac{1}{x} x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

c) $e^x \ln x$

$$(e^x \ln x)' = (e^x)' \ln x + e^x (\ln x)' = e^x \ln x + e^x \frac{1}{x} = e^x \left(\ln x + \frac{1}{x} \right)$$

C Derivative of $\ln f(x)$

Using (1) and the chain rule:

$$[\ln f(x)]' = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \qquad (2)$$

Ex 3. Differentiate and simplify.

a) $\ln(x^3 + x^2)$

$$[\ln(x^3 + x^2)]' = \frac{(x^3 + x^2)'}{x^3 + x^2} = \frac{3x^2 + 2x}{x^3 + x^2} = \frac{3x + 2}{x^2 + x}$$

b) $\ln \frac{x-1}{x+1}$

$$\left[\ln \frac{x-1}{x+1}\right]' = \frac{x+1}{x-1} \left(\frac{x-1}{x+1} \right)' = \frac{x+1}{x-1} \frac{(1)(x+1) - (x-1)(1)}{(x+1)^2} = \frac{2}{x^2 - 1}$$

c) $\ln(e^x + e^{-x})$

$$[\ln(e^x + e^{-x})]' = \frac{(e^x + e^{-x})'}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$$

D Derivative of $\log_b x$

$$(\log_b x)' = \frac{1}{(\ln b)x}$$

$$\frac{d}{dx}\log_b x = \frac{1}{(\ln b)x} \qquad (3)$$

Proof:

$$(\log_b x)' = \left(\frac{\ln x}{\ln b}\right)' = \frac{1}{\ln b}(\ln x)' = \frac{1}{(\ln b)x}$$

Ex 4. Differentiate.

a) $\log x$

$$(\log x)' = \frac{1}{(\ln 10)x}$$

b) $x^2 \log_3 x$

$$(x^2 \log_3 x)' = 2x\log_3 x + x^2 \frac{1}{(\ln 3)x} = 2x\log_3 x + \frac{x}{\ln 3}$$

c) $\frac{\log x}{10^x}$

$$\left(\frac{\log x}{10^x}\right)' = \frac{(\log x)'10^x - (\log x)(10^x)'}{(10^x)^2}$$

$$= \frac{\frac{1}{(\ln 10)x}10^x - (\log x)(10^x)(\ln 10)}{(10^x)^2} = \frac{\frac{1}{(\ln 10)x} - (\log x)(\ln 10)}{10^x}$$

E Derivative of $\log_b f(x)$

Using (3) and the chain rule:

$$[\log_b f(x)]' = \frac{f'(x)}{(\ln b)f(x)}$$

$$\frac{d}{dx}\log_b f(x) = \frac{f'(x)}{(\ln b)f(x)} \qquad (4)$$

Ex 5. Differentiate.

a) $\log(x^2+1)$

$$[\log(x^2+1)]' = \frac{(x^2+1)'}{(\ln 10)(x^2+1)} = \frac{2x}{(\ln 10)(x^2+1)}$$

b) $\log_2(x^2 2^x)$

$$[\log_2(x^2 2^x)]' = \frac{(x^2 2^x)'}{(\ln 2)x^2 2^x} = \frac{2x(2^x) + x^2(2^x)(\ln 2)}{(\ln 2)x^2 2^x}$$

$$= \frac{2 + x\ln 2}{x\ln 2}$$

c) $\log\ln x$

$$(\log\ln x)' = \frac{(\ln x)'}{(\ln 10)\ln x} = \frac{1}{(\ln 10)x\ln x}$$

Ex 6. Find the equation of the tangent line to the curve $y = f(x) = e^{-x}\ln x$ at the point $P(1,0)$.

$$f'(x) = e^{-x}(-1)\ln x + e^{-x}\frac{1}{x}$$

$$m = f'(1) = \frac{1}{e}$$

$$y - 0 = \frac{1}{e}(x-1) \quad \Rightarrow \quad \therefore y = \frac{1}{e}x - \frac{1}{e}$$

Ex 7. Find local extrema points for $f(x) = \frac{\ln x}{x}$.

$$f'(x) = \frac{\frac{1}{x}x - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \ at\ x = e, \quad f(e) = \frac{1}{e}$$

x		e	
$f(x)$	↗	$1/e$	↘
$f'(x)$	+	0	-

Local maximum point: $(e, 1/e)$.

Ex 8. Find the inflection points for $f(x) = x^2 \ln x$.

$f'(x) = 2x\ln x + x^2 \frac{1}{x} = 2x\ln x + x$

$f''(x) = 2\ln x + 2x\frac{1}{x} + 1 = 2\ln x + 3$

$f''(x) = 0 \text{ when } \ln x = -3/2 \Rightarrow x = e^{-3/2} = \frac{1}{e\sqrt{e}}$

$f(e^{-3/2}) = (e^{-3/2})^2\left(-\frac{3}{2}\right) = -\frac{3}{2e^3}$

x		$1/(e\sqrt{e})$	
$f(x)$	$\cap$	$-3/(2e^3)$	$\cup$
$f''(x)$	-	0	+

The inflection point is: $\left(\frac{1}{e\sqrt{e}}, -\frac{3}{2e^3}\right)$.

Ex 9. Find the global extrema for $f(x) = \frac{\log x}{x}$ over $[1,10]$.

$$f'(x) = \frac{\frac{1}{(\ln 10)x}x - (\log x)(1)}{x^2} = \frac{\frac{1}{\ln 10} - \log x}{x^2}$$

$f'(x) = 0 \text{ when } \log x = 1/\ln 10 \Rightarrow x = 10^{1/\ln 10} \cong 2.718$

$f(10^{1/\ln 10}) = \frac{1/\ln 10}{10^{1/\ln 10}} \cong 0.1598$

$f(1) = 0$

$f(10) = 0.1$

The global minimum point is $(1,0)$.

The global maximum point is $(2.718, 0.1598)$.

Ex 10. Differentiate.

a) $y = x^x$

$y' = (x^x)' = (e^{x\ln x})' = (e^{x\ln x})(x\ln x)' = (x^x)(\ln x + 1)$

b) $y = \ln|x|$

$$y = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$y' = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-x)' = \frac{1}{x}, & x < 0 \end{cases}$$

$\therefore y' = \frac{1}{x}, x \neq 0$

Reading: Nelson Textbook, Pages 571-574
Homework: Nelson Textbook: Page 575 #3ef, 4ace, 5ab, 6abc, 9a, 10, 11

Reading: Nelson Textbook, Pages 576-577
Homework: Nelson Textbook: Page 578 #1ad, 2cf, 3a, 4bd, 5, 8, 10

