5.4 5.5 Derivative of Trigonometric Functions

A Review of Trigonometric Functions

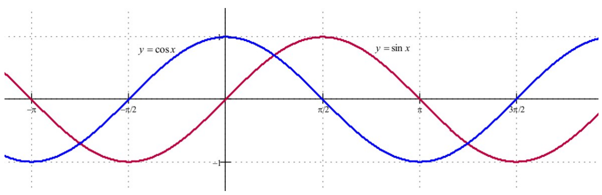
$\sin x : R \to [-1,1]$ $\quad \sin(x+2\pi) = \sin x$

$\cos x : R \to [-1,1]$ $\quad \cos(x+2\pi) = \cos x$

$\sin(x+\pi/2) = \cos x$ $\quad \sin(x+\pi) = -\sin x$

$\sin(2x) = 2\sin x \cos x$ $\quad \cos(2x) = \cos^2 x - \sin^2 x$

$\tan x = \dfrac{\sin x}{\cos x}$ $\quad \tan x : R \setminus \left\{\dfrac{\pi}{2} + n\pi, n \in Z\right\} \to R$



Ex 1. Compute the following limits.

a) $\lim_{x\to\infty} \sin x$ (use the graph of $\sin x$)

$\lim_{x\to\infty} \sin x$ DNE (Does Not Exist)

b) $\lim_{x\to\pi/2} \tan x$

$\lim_{x\to\pi/2} \tan x = \infty$

Ex 2. Use technology to evaluate each limit.

a) $\lim_{x\to 0} \dfrac{\sin x}{x}$

$\dfrac{\sin 0.001}{0.001} \cong 0.9999998333 \Rightarrow \therefore \lim_{x\to 0} \dfrac{\sin x}{x} = 1$ (*)

b) $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$

$\dfrac{\cos 0.001 - 1}{0.001} \cong -0.0005 \Rightarrow \therefore \lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$ (**)

B Derivative of $\sin x$

$$(\sin x)' = \cos x \quad (1)$$

Proof:

$(\sin x)' = \lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0} \dfrac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

$= \sin x \lim_{h\to 0} \dfrac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \dfrac{\sin h}{h}$

Use now the limits (*) and (**):

$(\sin x)' = \sin x(0) + \cos x(1) = \cos x$

Ex 3. Differentiate.

a) $x^3 \sin x$

$(x^3 \sin x)' = (x^3)' \sin x + x^3 (\sin x)' = 3x^2 \sin x + x^3 \cos x$

b) $\dfrac{e^x}{\sin x}$

$\left(\dfrac{e^x}{\sin x}\right)' = \dfrac{e^x \sin x - e^x \cos x}{\sin^2 x} = \dfrac{e^x (\sin x - \cos x)}{\sin^2 x}$

c) $\ln \sin x$

$(\ln \sin x)' = \dfrac{(\sin x)'}{\sin x} = \dfrac{\cos x}{\sin x}$

C Derivative of $\sin f(x)$

Using (1) and chain rule:

$$[\sin f(x)]' = (\cos f(x)) f'(x) \quad (2)$$

Ex 5. Differentiate.

a) $\sin x^2$

$\sin x^2 = (\cos x^2)(x^2)' = 2x \cos x^2$

b) $\sin e^x$

$(\sin e^x)' = (\cos e^x)(e^x)' = e^x \cos e^x$

c) $\sin \dfrac{1}{\ln x}$

$\left[\sin \dfrac{1}{\ln x}\right]' = \left(\cos \dfrac{1}{\ln x}\right)[(\ln x)^{-1}]'$

$= \left(\cos \dfrac{1}{\ln x}\right)(-1)(\ln x)^{-2}(\ln x)' = -\dfrac{\cos \dfrac{1}{\ln x}}{x \ln^2 x}$

D Derivative of $\cos x$

$$(\cos x)' = -\sin x \quad (3)$$

Proof:

$(\cos x)' = [\sin(x+\pi/2)]' = [\cos(x+\pi/2)](x+\pi/2)'$

$= -\sin x$

Ex 5. Differentiate.

a) $e^{-x}\cos x$

$(e^{-x}\cos x)' = (e^{-x})'\cos x + e^{-x}(\cos x)' = -e^{-x}\cos x + e^{-x}(-\sin x)$

$= -e^{-x}(\cos x + \sin x)$

b) $\cos^4 x$

$[\cos^4 x]' = 4\cos^3 x(\cos x)' = -4\cos^3 x \sin x$

E Derivative of $\cos f(x)$

Using (3) and chain rule:

$$[\cos f(x)]' = -[\sin f(x)]f'(x) \quad (4)$$

Ex 6. Differentiate.

a) $\cos(\sin x)$

$[\cos(\sin x)]' = [-\sin(\sin x)](\sin x)' = -(\cos x)\sin(\sin x)$

b) $\cos\frac{1}{x^2}$

$\left(\cos\frac{1}{x^2}\right)' = -\left(\sin\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)' = \frac{2}{x^3}\sin\frac{1}{x^2}$

c) $\cos(e^{x^2})$

$[\cos(e^{x^2})]' = -[\sin(e^{x^2})](e^{x^2})' = -[\sin(e^{x^2})](e^{x^2})(2x)$

$= -2xe^{x^2}\sin(e^{x^2})$

F Derivative of $\tan x$

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$

$$[\tan f(x)]' = \frac{f'(x)}{\cos^2 f(x)} = \sec^2 f(x) f'(x)$$

Proof:

$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x}$

$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

Ex 7. Differentiate.

a) $x^3 \tan x$

$(x^3\tan x)' = (x^3)'\tan x + x^3(\tan x)'$

$= 3x^2\tan x + \frac{x^3}{\cos^2 x}$

b) $\cot x$

$(\cot x)' = [(\tan x)^{-1}]' = -(\tan x)^{-2}(\tan x)'$

$= -\frac{1}{\tan^2 x}\frac{1}{\cos^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$

c) $\tan\sqrt{x^2+1}$

$(\tan\sqrt{x^2+1})' = \frac{(\sqrt{x^2+1})'}{\cos^2\sqrt{x^2+1}} = \frac{1}{\cos^2\sqrt{x^2+1}}\frac{2x}{2\sqrt{x^2+1}}$

$= \frac{x}{\sqrt{x^2+1}\cos^2\sqrt{x^2+1}}$

Ex 8. Find the equation of the tangent line to the graph of $y = f(x) = \sin x$ at $(\pi/4, 1/\sqrt{2})$. $f'(x) = \cos x$ $m = \cos(\pi/4) = 1/\sqrt{2}$ $y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$ $\therefore y = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$	Ex 9. Find the points of inflection for $f(x) = \cos^2 x$ over the interval $[0, \pi]$. $f'(x) = 2\cos x(\cos x)' = -2\sin x \cos x$ $f''(x) = -2(\sin x)'\cos x - 2\sin x(\cos x)'$ $= -2\cos^2 x + 2\sin^2 x$ $f''(x) = 0 \Rightarrow \sin^2 x = \cos^2 x \Rightarrow \tan^2 x = 1 \Rightarrow \tan x = \pm 1$ $f''(x) = 0 \; at \; \frac{\pi}{4}, \frac{3\pi}{4}$ The inflection points are: $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ and $\left(\frac{3\pi}{4}, \frac{1}{2}\right)$
Ex 10. Find the global extrema for $f(x) = x + \cos x, \quad 0 \le x \le 2\pi$. $f'(x) = 1 - \sin x$ $f'(x) = 0 \; when \; \sin x = 1 \Rightarrow x = \pi/2$ $f(\pi/2) = \pi/2$ $f(0) = 1$ $f(2\pi) = 2\pi + 1$ The global maximum point is $(2\pi, 2\pi + 1)$. The global minimum point is $(0,1)$.	Ex 11. Consider the following position function: $s(t) = 5\sin\left(\frac{\pi t}{10}\right)$. Prove that: $a(t) + \frac{\pi^2}{100}s(t) = 0$. $v(t) = s'(t) = 5\cos\left(\frac{\pi t}{10}\right)\left(\frac{\pi t}{10}\right)' = \frac{\pi}{2}\cos\left(\frac{\pi t}{10}\right)$ $a(t) = v'(t) = \frac{\pi}{2}(-1)\sin\left(\frac{\pi t}{10}\right)\left(\frac{\pi t}{10}\right)' = -\frac{\pi^2}{20}\sin\left(\frac{\pi t}{10}\right)$ $a(t) + \frac{\pi^2}{100}s(t) = -\frac{\pi^2}{20}\sin\left(\frac{\pi t}{10}\right) + \frac{\pi^2}{100}(5)\sin\left(\frac{\pi t}{10}\right) = 0$

Reading: Nelson Textbook, Pages 250-256
Homework: Nelson Textbook: Page 256 #1cfj, 2acd, 3b, 5ad, 6c, 11, 12, 14

Reading: Nelson Textbook, Pages 258-259
Homework: Nelson Textbook: Page 260 #1ade, 2b, 3abf, 4a, 6, 7, 8

