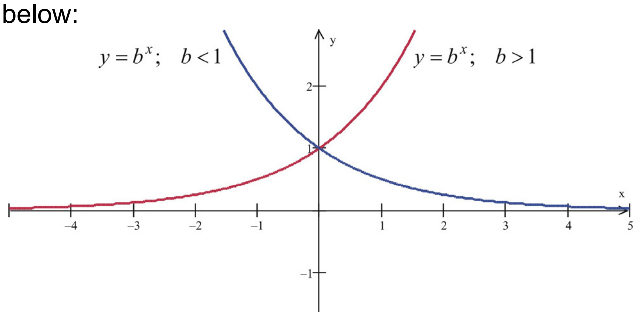
5.1 5.2 Derivative of Exponential Function

A Review of Exponential Functions

The exponential function is defined as:

$$y = f(x) = b^x; \quad b > 0, b \neq 1$$

The graph of the exponential function is represented below:



The x-axis ($y = 0$) is a horizontal asymptote.

Ex 1. Use the graph of the exponential function to evaluate each limit.

a) $\lim_{x\to\infty} 2^x$

$$\lim_{x\to\infty} 2^x = \infty$$

b) $\lim_{x\to-\infty} 2^x$

$$\lim_{x\to-\infty} 2^x = 0$$

c) $\lim_{x\to\infty} 0.2^x$

$$\lim_{x\to\infty} 0.2^x = 0$$

d) $\lim_{x\to-\infty} 0.2^x$

$$\lim_{x\to-\infty} 0.2^x = \infty$$

B Number e

The number e is defined by:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad (1)$$

which can be written also as:

$$e = \lim_{u\to 0}(1+u)^{\frac{1}{u}} \qquad (2)$$

Ex 2. Estimate the number e using formula (1) and by taking $n = 100000$.

$$e \cong \left(1+\frac{1}{100000}\right)^{100000} \cong 2.7183$$

C Exponential Function

The exponential function e^x may be evaluate using the limit:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad (3)$$

Proof:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x = \left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x =$$

$$= \left(\lim_{u\to 0}(1+u)^{\frac{1}{u}}\right)^x = e^x$$

Ex 3. Estimate $\sqrt{e}$ using formula (2) and by taking $n = 100000$.

$$\sqrt{e} = e^{1/2} \cong \left(1+\frac{1/2}{100000}\right)^{100000} \cong 1.6487$$

D Derivative of e^x

$$(e^x)' = e^x$$

$$\frac{d}{dx}e^x = e^x \qquad (4)$$

Proof:

$$\left(1+\frac{x}{n}\right)^n \to e^x \quad \Rightarrow \quad n\left(1+\frac{x}{n}\right)^{n-1}\left(\frac{1}{n}\right) \to (e^x)'$$

$$\left(\left(1+\frac{x}{n}\right)^n\right)^{\frac{n-1}{n}} \to \left(1+\frac{x}{n}\right)^n \to e^x \quad \Rightarrow \quad \therefore (e^x)' = e^x$$

Ex 4. Differentiate and simplify.

a) x^2e^x

$$(x^2e^x)' = (x^2)'e^x + x^2(e^x)' = 2xe^x + x^2e^x = xe^x(2+x)$$

b) $e^{x/2}$

$$(e^{x/2})' = (\sqrt{e^x})' = \frac{1}{2\sqrt{e^x}}e^x = \frac{1}{2}e^{x/2}$$

c) e^{-x}

$$(e^{-x})' = [(e^x)^{-1}]' = (-1)(e^x)^{-2}(e^x) = -e^{-2x+x} = -e^{-x}$$

E Derivative of $e^{f(x)}$
Using (4) and the chain rule:

$$(e^{f(x)})' = e^{f(x)} f'(x)$$
$$\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x) \qquad (5)$$

Ex 5. Differentiate.

a) e^{-3x}

$(e^{-3x})' = e^{-3x}(-3x)' = -3e^{-3x}$

b) e^{-1/x^2}

$(e^{-1/x^2})' = e^{-1/x^2}(-x^{-2})' = \frac{2e^{-1/x^2}}{x^3}$

c) $e^{\sqrt{x^2+1}}$

$(e^{\sqrt{x^2+1}})' = e^{\sqrt{x^2+1}}(\sqrt{x^2+1})' = \frac{(2x)e^{\sqrt{x^2+1}}}{2\sqrt{x^2+1}} = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

Ex 6. The hyperbolic functions are defined by:

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$.

Prove that:

a) $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1$

b) $(\sinh x)' = \cosh x$

$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{(e^x - e^{-x})'}{2} = \frac{e^x - e^{-x}(-1)}{2}$

$= \frac{e^x + e^{-x}}{2} = \cosh x$

c) $(\cosh x)' = \sinh x$

$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{(e^x + e^{-x})'}{2} = \frac{e^x + e^{-x}(-1)}{2}$

$= \frac{e^x - e^{-x}}{2} = \sinh x$

d) $(\tanh x)' = \frac{1}{\cosh^2 x}$

$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\sinh x)'\cosh x - \sinh x(\cosh x)'}{\cosh^2 x}$

$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

F Derivative of $b^x, \quad b > 0, b \neq 1$

$$(b^x)' = (\ln b) b^x$$
$$\frac{d}{dx} b^x = (\ln b) b^x \qquad (6)$$

Proof:

$(b^x)' = (e^{x \ln b})' = e^{x \ln b}(\ln b) = (\ln b) b^x$

Ex 7. Differentiate.

a) 3^x

$(3^x)' = (\ln 3) 3^x$

b) $x^2 2^x$

$(x^2 2^x)' = (x^2)'(2^x) + (x^2)(2^x)' = 2x(2^x) + x^2(2^x)(\ln 2)$

$= x(2^x)(2 + x\ln 2)$

c) $(4^x + x^4)^3$

$[(4^x + x^4)^3]' = 3(4^x + x^4)^2(4^x + x^4)'$

$= 3(4^x + x^4)^2[(\ln 4)4^x + 4x^3]$

G Derivative of $b^{f(x)}$
Using (6) and the chain rule:

$$(b^{f(x)})' = (\ln b) b^{f(x)} f'(x)$$
$$\frac{d}{dx} b^{f(x)} = (\ln b) b^{f(x)} f'(x) \qquad (7)$$

Ex 8. Differentiate.

a) 2^{-x^3}

$(2^{-x^3})' = (\ln 2)(2^{-x^3})(-x^3)' = -3(\ln 2)(2^{-x^3})x^2$

b) $10^{\sqrt{e^x - x^2}}$

$$\left(10^{\sqrt{e^x-x^2}}\right)' = (\ln 10)\left(10^{\sqrt{e^x-x^2}}\right)(\sqrt{e^x-x^2})'$$

$$= (\ln 10)\left(10^{\sqrt{e^x-x^2}}\right)\frac{e^x-2x}{2\sqrt{e^x-x^2}}$$

Ex 9. Find the equation of the tangent line to the graph of $y = f(x) = x(2^{-x})$ at $(0,0)$.

$f'(x) = 2^{-x} - x(\ln 2)(2^{-x})$

$m = f'(0) = 2^{-0} - (0)(\ln 2)(2^{-0}) = 1$

$y - 0 = 1(x - 0) \quad \Rightarrow \quad \therefore y = x$

Ex 10. Find the local extrema for $y = f(x) = x^2 e^{-x^2}$.

$f'(x) = 2xe^{-x^2} + x^2 e^{-x^2}(-2x) = 2xe^{-x^2}(1 - x^2)$

$f'(x) = 0 \ at \ x = 0, -1, +1$

$f(0) = 0, \quad f(-1) = e^{-1}, \quad f(1) = e^{-1}$

x		-1		0		1	
$f(x)$	↗	$1/e$	↘	0	↗	$1/e$	↘
$f'(x)$	+	0	-	0	+	0	-

Local maximum points: $(\pm 1, 1/e)$.

Local minimum point: $(0,0)$.

Ex 11. Find the points of inflection for $y = f(x) = e^{-x^2}$.

$f'(x) = -2xe^{-x^2}$

$f''(x) = -2e^{-x^2} - 2x(-2xe^{-x^2})$

$= 2e^{-x^2}(2x^2 - 1)$

$f''(x) = 0 \ at \ x = \pm 1/\sqrt{2}$

$f(\pm 1/\sqrt{2}) = e^{-1/2} = 1/\sqrt{e}$

x		$-1/\sqrt{2}$		$1/\sqrt{2}$	
$f(x)$	$\cup$	$1/\sqrt{e}$	$\cap$	$1/\sqrt{e}$	$\cup$
$f''(x)$	+	0	-	0	+

The inflection points are $(\pm 1/\sqrt{2}, 1/\sqrt{e})$.

Ex 12. Find the global extrema for $f(x) = x^3 10^{-x}$ over $[-1,2]$.

$f'(x) = 3x^2 10^{-x} - x^3(\ln 10)(10^{-x}) = x^2 10^{-x}[3 - (\ln 10)x]$

$f'(x) = 0 \ at \ x = 0, 3/\ln 10 \cong 1.303$

$f(0) = 0$

$f(3/\ln 10) = (3/\ln 10)^3 10^{-(3/\ln 10)} \cong 0.110$

$f(-1) = -10$

$f(2) = 8(10^{-2}) = 0.08$

The global maximum point is: $(1.303, 0.110)$.

The global minimum point is $(-1, -10)$.

Reading: Nelson Textbook, Pages 227-232
Homework: Nelson Textbook: Page 232 #2ef, 3cdf, 4abc, 5a, 8, 10a, 13, 16, 17

Reading: Nelson Textbook, Pages 235-239
Homework: Nelson Textbook: Page 240 #1bd, 2abcd, 3, 4, 6, 8, 9