4.5 An Algorithm for Curve Sketching

A Algorithm for Curve Sketching

1. Domain

⇨ $denominator \neq 0$ (rational functions)
⇨ $radicand \geq 0$ (even roots)
⇨ $logarithmic\ argument > 0$ (logarithmic functions)

2. Intercepts

⇨ $f(x) = 0$ (x-intercepts or zeros)
⇨ $numerator = 0$ (for rational functions)
⇨ $y - \text{int} = f(0)$ (if exists)

3. Symmetry

⇨ $f(-x) = f(x)$ (even functions are symmetric about the y-axis)
⇨ $f(-x) = -f(x)$ (odd functions are symmetric about the origin)
⇨ $f(x+T) = f(x)$ (periodic functions have cycles)

4. Asymptotes

⇨ compute $\lim_{x \to \pm\infty} f(x)$ (horizontal asymptote)
⇨ compute $\lim_{x \to a} f(x)$ (vertical asymptote where a is a zero of the denominator but not of the numerator)
⇨ compute long division (to find the oblique asymptotes for rational functions)

5. First Derivative

⇨ compute $f'(x)$
⇨ find critical points ($f'(x) = 0$ or $f'(x)$ DNE)
⇨ create the sign chart for $f'(x)$
⇨ find intervals of increase/decrease
⇨ find the local extrema (using first derivative test) and global extrema (if function is defined on a closed interval)

6. Second Derivative

⇨ compute $f''(x)$
⇨ find points where $f''(x) = 0$ or $f''(x)$ DNE
⇨ create the sign chart for $f''(x)$
⇨ find points of inflection
⇨ find intervals of concavity upward/downward
⇨ check the local extrema using the second derivative test (if necessary)

7. Curve Sketching

⇨ use broken lines to draw the asymptotes
⇨ plot x- and y- intercepts, extrema, and inflection points
⇨ draw the curve near the asymptotes
⇨ sketch the curve

Ex 1. Sketch the graph for $y = f(x) = 3x^5 - 5x^3$.

Domain: $x \in R$.

Intercepts: $f(x) = 0 \Rightarrow x = 0 \text{ or } x = \pm\sqrt{5/3}$, $f(0) = 0$

Symmetry:

$f(-x) = 3(-x)^5 - 5(-x)^3 = -3x^5 + 5x^3 = -f(x)$

The function $y = f(x)$ is odd.

Asymptotes: none

First Derivative:

$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$

$f'(x) = 0 \Rightarrow x = 0 \text{ or } x = \pm 1$

$f(0) = 0$, $f(-1) = -3 + 5 = 2$, $f(1) = 3 - 5 = -2$

x		-1		0		1	
$f(x)$	↗	2	↘	0	↘	–2	↗
$f'(x)$	+	0	-	0	-	0	+

$(-1,2)$ is a local maximum point.

$(1,-2)$ is a local minimum point.

Second Derivative:

$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

$f''(x) = 0 \Rightarrow x = 0 \text{ or } x = \pm 1/\sqrt{2}$

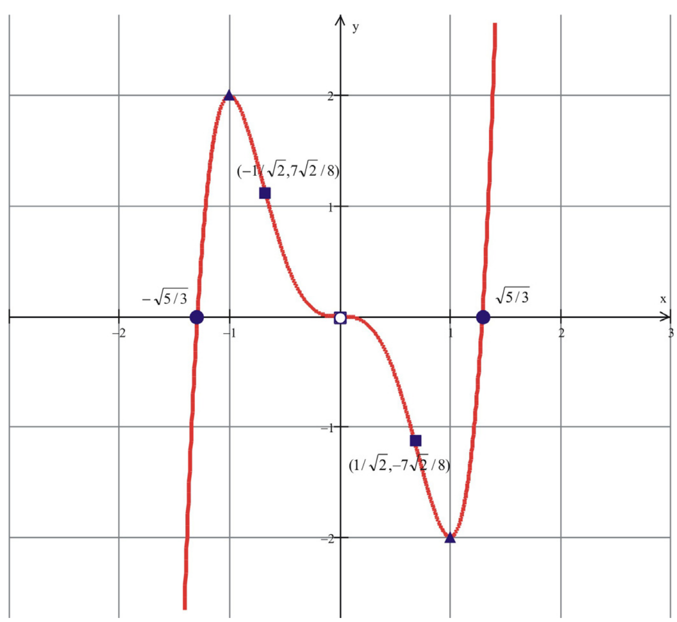
$f(0) = 0$, $f(-1/\sqrt{2}) = (-1/\sqrt{2})^3[3(1/2) - 5) = 7\sqrt{2}/8 \cong 1.24$

$f(1/\sqrt{2}) = -7\sqrt{2}/8 \cong -1.24$

x		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	
$f(x)$	∩	$7\sqrt{2}/8$	∪	0	∩	$-7\sqrt{2}/8$	∪
$f''(x)$	-	0	+	0	-	0	+

$(-1/\sqrt{2}, 7\sqrt{2}/8)$ and $(1/\sqrt{2}, -7\sqrt{2}/8)$ are points of inflection.

Curve Sketching:



Ex 2. Sketch the graph for $y = f(x) = x^3 - 6x^2 + 9x + 1$.

Domain: $x \in R$.

Intercepts:

$f(-1) = -1 - 6 - 9 + 1 = -15$

$f(1) = 1 - 6 + 9 + 1 = 5$

$f(0) = 1$

There are no rational zeros.

Symmetry:

$f(-x) = (-x)^3 - 6(-x)^2 + 9(-x) + 1$

$= -x^3 - 6x^2 - 9x + 1$

$f(-x) \neq f(x), \quad f(-x) \neq -f(x)$

The function $y = f(x)$ is neither odd nor even.

Asymptotes: none

First Derivative:

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

$f'(x) = 0 \Rightarrow x = 1 \ or \ x = 3$

$f(1) = 5, \quad f(3) = 27 - 54 + 27 + 1 = 1$

x		1		3	
$f(x)$	↗	5	↘	1	↗
$f'(x)$	+	0	-	0	+

$(1,5)$ is a local maximum point.

$(3,1)$ is a local minimum point.

Second Derivative:

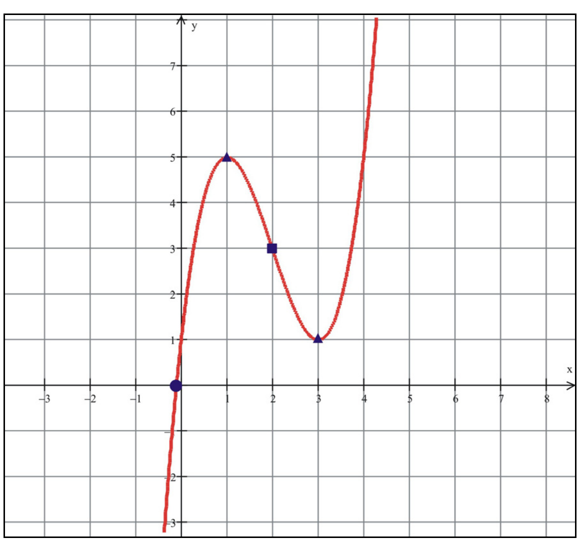
$f''(x) = 6x - 12 = 6(x-2)$

$f''(x) = 0 \Rightarrow x = 2, \quad f(2) = 8 - 24 + 18 + 1 = 3$

x		2	
$f(x)$	$\cap$	3	$\cup$
$f''(x)$	-	0	+

$(2,3)$ is a point of inflection.

Curve Sketching:



Ex 3. Sketch the graph for $y = f(x) = \dfrac{4x}{x^2+1}$.

Domain: $x \in R$.

Intercepts: $f(x) = 0 \Rightarrow x = 0, \quad f(0) = 0$

Symmetry:

$$f(-x) = \frac{4(-x)}{(-x)^2+1} = -\frac{4x}{x^2+1} = -f(x)$$

The function $y = f(x)$ is odd.

Asymptotes: $y = 0$ is a horizontal asymptote.

First Derivative:

$$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2} = \frac{4 - 4x^2}{(x^2+1)^2} = \frac{4(1-x^2)}{(x^2+1)^2}$$

$$f'(x) = 0 \Rightarrow x = \pm 1, \quad f(\pm 1) = \frac{4(\pm 1)}{(\pm 1)^2 + 1} = \frac{\pm 4}{2} = \pm 2$$

x		-1		1	
$f(x)$	↘	-2	↗	2	↘
$f'(x)$	-	0	+	0	-

$(1,2)$ is a local maximum point.

$(-1,-2)$ is a local minimum point.

Second Derivative:

$$f''(x) = \frac{-8x(x^2+1)^2 - 4(1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{-8x^3 - 8x - 16x + 16x^3}{(x^2+1)^3} = \frac{8x^3 - 24x}{(x^2+1)^3} = \frac{8x(x^2-3)}{(x^2+1)^3}$$

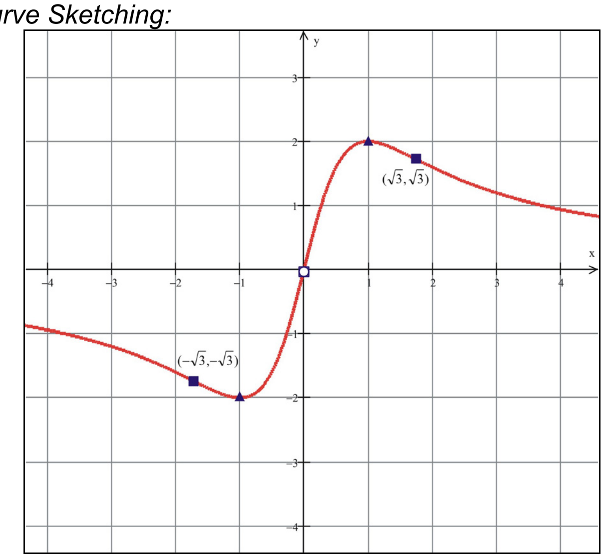
$f''(x) = 0 \Rightarrow x = 0 \ or \ x = \pm\sqrt{3}$

$$f(0) = 0, \quad f(\pm\sqrt{3}) = \frac{\pm 4\sqrt{3}}{(\pm\sqrt{3})^2 + 1} = \pm\sqrt{3}$$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$f(x)$	$\cap$	$-\sqrt{3}$	$\cup$	0	$\cap$	$\sqrt{3}$	$\cup$
$f''(x)$	-	0	+	0	-	0	+

$(-\sqrt{3}, -\sqrt{3})$, $(0,0)$, and $(\sqrt{3}, \sqrt{3})$ are points of inflection.

Curve Sketching:



Ex 4. Sketch the graph for $y = f(x) = \dfrac{x^2}{x-1}$.

Domain: $x \in R \setminus \{1\}$.

Intercepts: $f(x) = 0 \Rightarrow x = 0, \quad f(0) = 0$

Symmetry:

$$f(-x) = \frac{(-x)^2}{-x-1} = -\frac{x^2}{x+1}$$

$f(-x) \neq f(x), \quad f(-x) \neq -f(x)$

The function $y = f(x)$ is neither odd nor even.

Asymptotes:

$$f(x) = \frac{x^2-1+1}{x-1} = x+1+\frac{1}{x-1}$$

$y = x+1$ is the equation of the oblique asymptote.

First Derivative:

$$f'(x) = \frac{(2x)(x-1)-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$f'(x) = 0 \Rightarrow x = 0 \text{ or } x = 2, \quad f'(x) \text{ DNE at } x = 1$

$f(0) = 0, \quad f(2) = 4, \ f(1) \text{ DNE}$

x		0		1		2	
$f(x)$	↗	0	↘	DNE	↘	4	↗
$f'(x)$	+	0	-	DNE	-	0	+

$(0,0)$ is a local maximum point.

$(1,4)$ is a local minimum point.

Second Derivative:

$$f''(x) = \frac{2(x-1)(x-1)^2 - x(x-2)(2)(x-1)}{(x-1)^4}$$

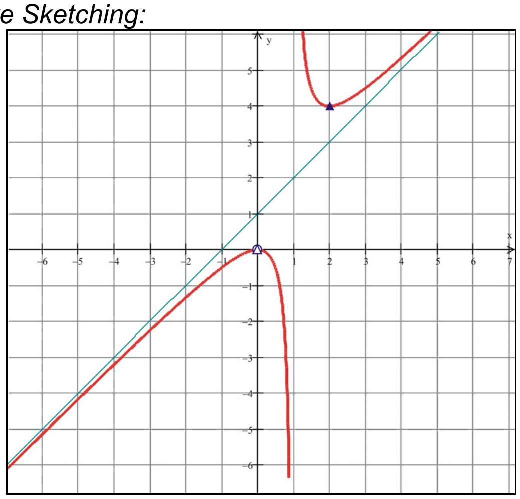
$$= \frac{2[(x-1)(x-1)-x(x-2)]}{(x-1)^3} = \frac{2}{(x-1)^3}$$

$f''(1)$ DNE

x		1	
$f(x)$	∩	DNE	∪
$f''(x)$	-	DNE	+

There are no inflection points.

Curve Sketching:



Ex 5. Sketch the graph for $y = f(x) = x(5-x)^{2/3}$.

Domain: $x \in R$.

Intercepts: $f(x) = 0 \Rightarrow x = 0 \text{ or } x = 5, \quad f(0) = 0$

Symmetry:

$f(-x) = -x(5+x)^{2/3}, \quad f(-x) \neq f(x), \quad f(-x) \neq -f(x)$

The function $y = f(x)$ is neither odd nor even.

Asymptotes: The function behaves at infinity as $x^{5/3}$.
There is no asymptote.

First Derivative:

$$f'(x) = (5-x)^{2/3} + x\frac{2}{3}(5-x)^{-1/3}(-1)$$

$$= \frac{3}{3}(5-x)^{-1/3}(5-x)^{1/3}(5-x)^{2/3} + x\frac{2}{3}(5-x)^{-1/3}(-1)$$

$$= \frac{3(5-x)-2x}{3(5-x)^{1/3}} = \frac{15-5x}{3(5-x)^{1/3}} = \frac{5(3-x)}{3(5-x)^{1/3}}$$

$f'(x) = 0 \text{ at } x = 3, \quad f(3) = 3(5-3)^{2/3} = 3\sqrt[3]{4} \cong 4.76$

$f'(x) \text{ DNE at } x = 0, \quad f(0) = 0$

x		3		5	
$f(x)$	↗	$3\sqrt[3]{4}$	↘	0	↗
$f'(x)$	+	0	-	DNE	+

$(3, 3\sqrt[3]{4})$ is a local maximum point.

$(5,0)$ is a local minimum point.

Second Derivative:

$$f''(x) = \frac{5}{3}[(-1)(5-x)^{-1/3} + (3-x)(-1/3)(5-x)^{-4/3}(-1)]$$

$$= \frac{5}{3}\left[-\frac{3}{3}(5-x)^{-1/3}(5-x)^{-4/3}(5-x)^{4/3} + \frac{1}{3}(3-x)(5-x)^{-4/3}\right]$$

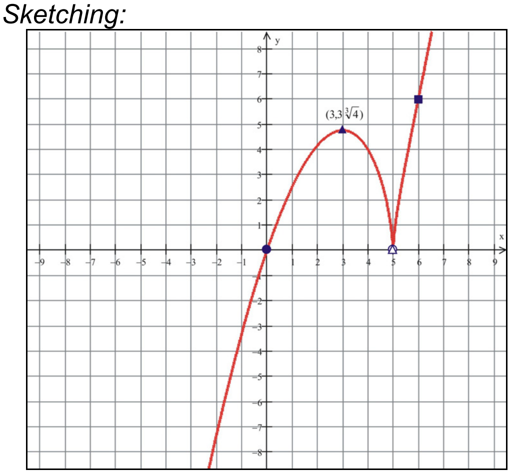
$$= \frac{5}{3}\frac{-3(5-x)+(3-x)}{3(5-x)^{4/3}} = \frac{5}{3}\frac{2x-12}{3(5-x)^{4/3}} = \frac{10(x-6)}{9(5-x)^{4/3}}$$

$f''(x) = 0 \text{ at } x = 6, \quad f(6) = 6(5-6)^{2/3} = 6$

x		5		6	
$f(x)$	∩	0	∩	6	∪
$f''(x)$	-	DNE	-	0	+

$(6,6)$ is a point of inflection.

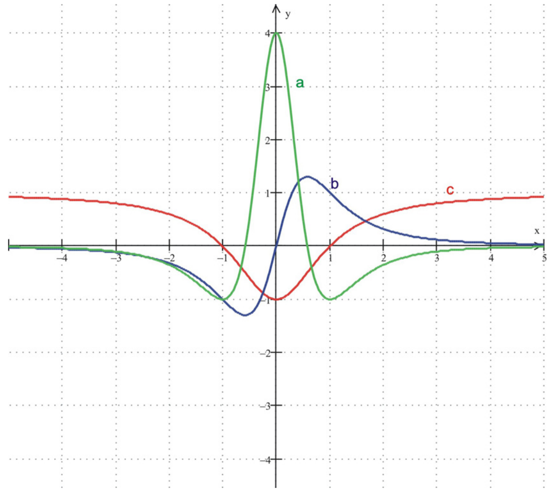
Curve Sketching:



B Link between a function and its derivative
Consider a double differentiable function $y = f(x)$ ($f'(x)$ and $f''(x)$ exist). Then:

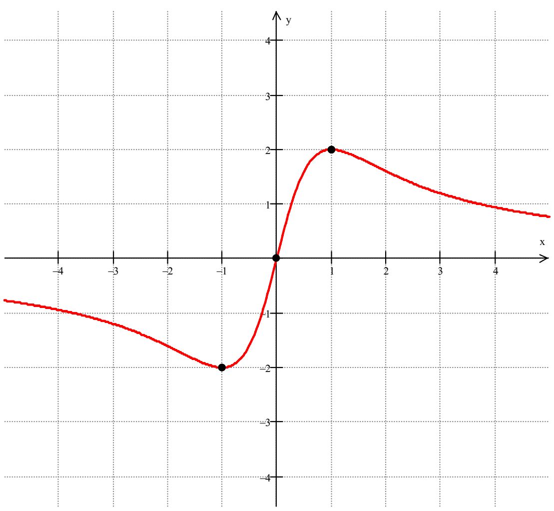
1. $f'(x)$ is the slope of the tangent at $P(x, f(x))$.
2. If $f'(x) = 0$, then $P(x, f(x))$ is a local extrema and tangent is horizontal.
3. If $f'(x) > 0$, then the function $y = f(x)$ is increasing.
4. If $f'(x) < 0$, then the function $y = f(x)$ is decreasing.
5. If $f''(x) = 0$, then $f'(x)$ has a local extrema and $y = f(x)$ has an inflection point.
6. If $f''(x) > 0$, then $f'(x)$ is increasing and $y = f(x)$ is concave upward.
7. If $f''(x) < 0$, then $f'(x)$ is decreasing and $y = f(x)$ is concave downward.

Ex 6. The graphs of a function and its first and second derivatives are represented on the same grid. Identify each of them.



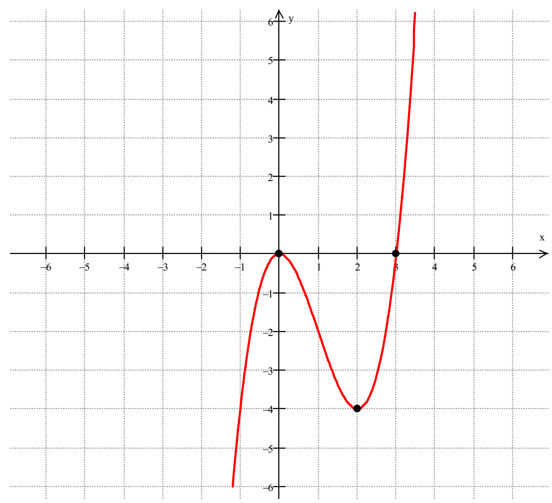
$c \Rightarrow f(x), \quad b \Rightarrow f'(x), \quad a \Rightarrow f''(x)$

Ex 7. In the next figure is given the graph of a function $y = f(x)$.



a) Find the intervals where $f'(x)$ is positive and negative.
$f'(x) > 0$ where the function is increasing: $(-1,1)$.
$f'(x) < 0$ where the function is decreasing: $(-\infty,-1)$ or $(1,\infty)$.
b) *Estimate* intervals where $f''(x)$ is positive and negative.
$f''(x) > 0$ where the graph is concave upward: $(-2,0)$ or $(2,\infty)$ (approximate).
$f''(x) < 0$ where the graph is concave downward: $(-\infty,-2)$ or $(0,2)$ (approximate).

Ex 8. In the next figure is given the graph of the derivative $f'(x)$ of a function $f(x)$.



a) Find intervals where the function $f(x)$ is increasing or decreasing.
The function $f(x)$ is increasing where $f'(x) > 0$: $(3,\infty)$.
The function $f(x)$ is decreasing where $f'(x) < 0$: $(-\infty,0)$ or $(0,3)$.
b) Find intervals where the graph of $f(x)$ is concave upward or downward.
The graph of $f(x)$ is concave upward where $f'(x)$ is increasing: $(-\infty,0)$ or $(2,\infty)$.
The graph of $f(x)$ is concave downward where $f'(x)$ is decreasing: $(0,2)$.

Reading: Nelson Textbook, Pages 207-212
Homework: Nelson Textbook: Page 213 #4begij, 6, 7b, 9

