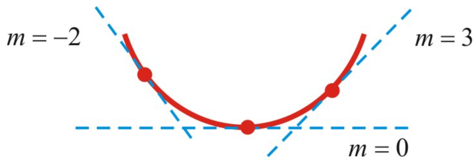
4.4 Concavity and Points of Inflection

A Concavity

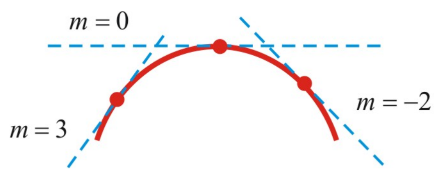
The graph of a function has a *concavity upward* if:

- Graph lies above all its tangents
- Tangents rotate counter-clockwise
- Slope of tangent lines increases
- $f'(x)$ *increases* or $f''(x) > 0$



The graph of a function has a *concavity downward* if:

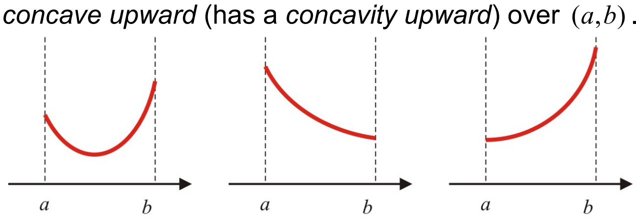
- Graph lies below all its tangents
- Tangents rotate clockwise
- Slope of tangent lines decreases
- $f'(x)$ *decreases* or $f''(x) < 0$



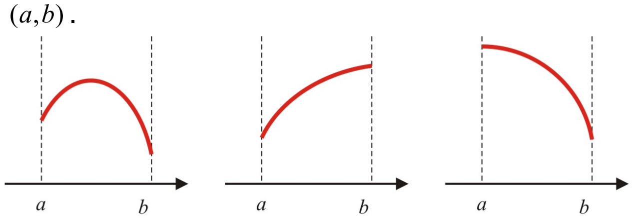
B Test for Concavity

Let f be a function *twice differentiable* ($f''(x)$ exists) over (a,b).

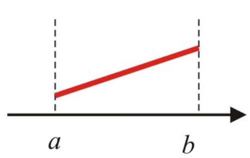
1. If $f''(x) > 0$ for all $x \in (a,b)$, then the graph of f is *concave upward* (has a *concavity upward*) over (a,b).



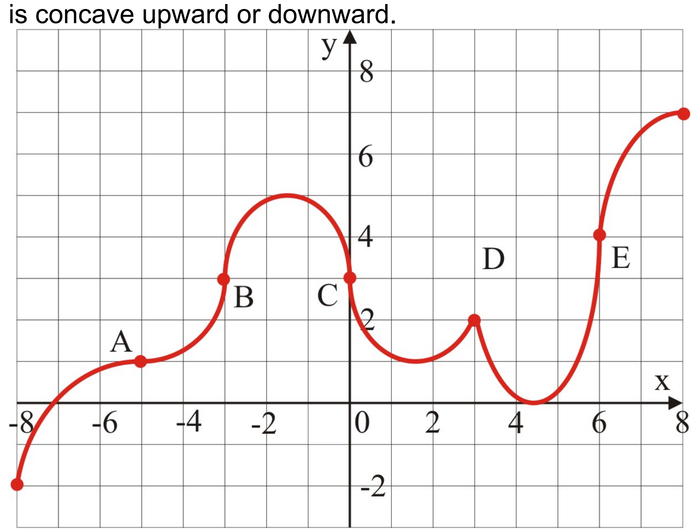
2. If $f''(x) < 0$ for all $x \in (a,b)$, then the graph of f is *concave downward* (has a *concavity downward*) over (a,b).



3. 1. If $f''(x) = 0$ for all $x \in (a,b)$, then the graph of f has *no concavity* over (a,b) ($f'(x) = const$; the graph is a straight line).



Ex 1. Find the intervals on which the graph, given below, is concave upward or downward.



The graph is concave upward over $(-5,-3)$, $(0,3)$, and $(3,6)$.

The graph is concave downward over $(-8,-5)$, $(-3,0)$, and $(6,8)$.

Ex 2. Find the intervals of concavity for $f(x) = x^4 - 2x^3$.

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x = 12x(x-1)$

$f''(x) = 0 \quad \Rightarrow \quad x = 0 \; or \; x = 1$

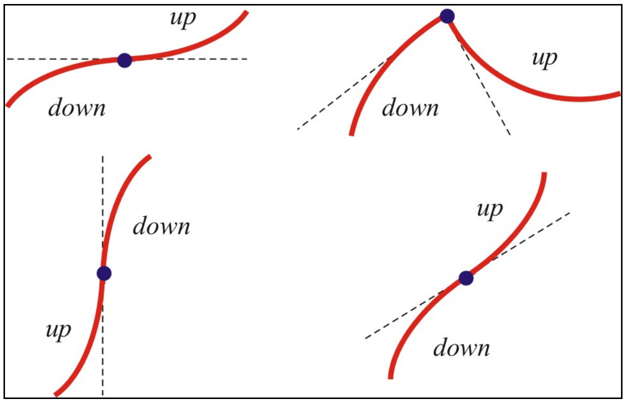
$f(0) = 0, \quad f(1) = -1$

x		0		1	
$f(x)$	$\cup$	0	$\cap$	-1	$\cup$
$f''(x)$	+	0	-	0	+

The graph is concave upward over $(-\infty,0)$ and $(1,\infty)$.

The graph is concave downward over $(0,1)$.

C Point of Inflection
A point $P(i, f(i))$ on the graph of $y = f(x)$ is called *point of inflection* if the concavity of the graph changes at P (from concave upward to concave downward or from concave downward to concave upward).



Ex 2. Use the function given at Ex. 1 to identify the points of inflection.

The points of inflection are $A(-5,1)$, $B(-3,3)$, $C(0,3)$, and $E(6,4)$.

Ex 3. Use the sign chart obtained at Ex 2. to identify the points of inflection for $f(x) = x^4 - 2x^3$.

x		0		1	
$f(x)$	$\cup$	0	$\cap$	-1	$\cup$
$f''(x)$	+	0	-	0	+

The points of inflection are $(0,0)$, and $(1,-1)$.

Ex 4. Find the intervals of concavity and the inflection points for $f(x) = \dfrac{x^2}{x^2+1}$.

$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$f''(x) = \frac{2(x^2+1)^2 - 2x(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{2x^2+2-8x^2}{(x^2+1)^3} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$$f''(x) = 0 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow \quad f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{1/3}{1/3+1} = \frac{1}{4}$$

x		$-1/\sqrt{3}$		$1/\sqrt{3}$	
$f(x)$	$\cap$	$1/4$	$\cup$	$1/4$	$\cap$
$f''(x)$	-	0	+	0	-

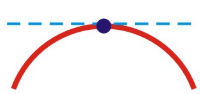
The graph is concave upward over $(-1/\sqrt{3}, 1/\sqrt{3})$.

The graph is concave downward over $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$.

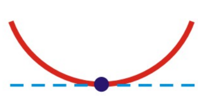
The points of inflection are $(-1/\sqrt{3}, 1/4)$, and $(1/\sqrt{3}, 1/4)$.

D Second Derivative Test
Let f be a *twice differentiable* function over an open interval containing the *critical number* c and $f'(c) = 0$ ($(c, f(c))$ is a *stationary* point).

1. If $f''(c) > 0$ then f has a *local minimum* at $x = c$.
2. If $f''(c) < 0$ then f has a *local maximum* at $x = c$.

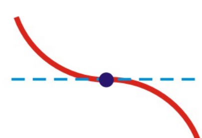


$f'(c) = 0$
$f''(c) < 0$
$(maximum\ point)$

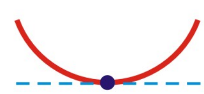


$f'(c) = 0$
$f''(c) > 0$
$(minimum\ point)$

3. If $f''(c) = 0$ then the function may have a local minimum, maximum, or neither (inconclusive case). Use the first derivative test to conclude.



$f'(c) = 0$
$f''(c) = 0$
$(no\ local\ extremum)$



$f'(c) = 0$
$f''(c) = 0$
$(minimum\ point)$

Ex 6. Use the second derivative test to find the local extrema. If the second derivative test is not conclusive (fails), then use the first derivative test to conclude.

a) $f(x)=(x-1)^3$

$f'(x)=3(x-1)^2$

$f'(x)=0 \; at \; x=1, \quad f(1)=0$

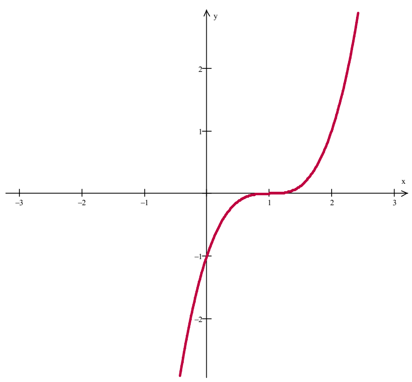
$f''(x)=6(x-1)$

$f''(1)=0 \quad (inconclusive\ case)$

Let use the first derivative test.

x		1	
$f(x)$	↗	0	↗
$f'(x)$	+	0	+

Therefore, is no local extrema at $(1,0)$. See the graph below.



b) $f(x)=(3-2x)^4$

$f'(x)=4(3-2x)^3(-2)=-8(3-2x)^3$

$f'(x)=0 \; at \; x=3/2, \quad f(3/2)=0$

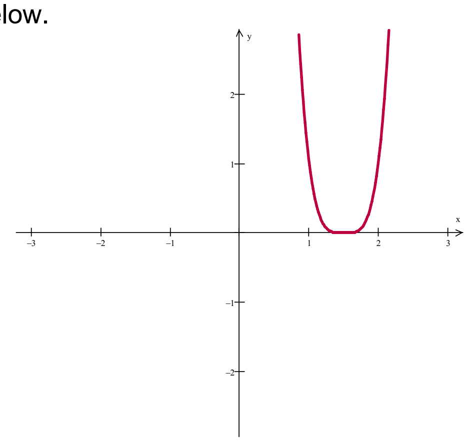
$f''(x)=-24(3-2x)^2(-2)$

$f''(3/2)=0 \quad (inconclusive\ case)$

Let use the first derivative test.

x		$3/2$	
$f(x)$	↘	0	↗
$f'(x)$	-	0	+

Therefore, there is local minimum at $(3/2,0)$. See the graph below.



Ex 5. Use the second derivative test to find the local extrema for $f(x)=\dfrac{x}{x^2+1}$.

$$f'(x)=\frac{(1)(x^2+1)-x(2x)}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$$

$$f'(x)=0 \; at \; x=\pm 1 \quad \Rightarrow f(\pm 1)=\frac{\pm 1}{2}$$

$$f''(x)=\frac{(-2x)(x^2+1)^2-(1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$=\frac{-2x^3-2x-4x+4x^3}{(x^2+1)^3}=\frac{2x^3-6x}{(x^2+1)^3}$$

$f''(-1)=\dfrac{-2+6}{8}>0 \quad \Rightarrow \quad \therefore (-1,-1/2)$ is a local minimum point.

$f''(1)=\dfrac{2-6}{8}<0 \quad \Rightarrow \quad \therefore (1,1/2)$ is a local maximum point.

Reading: Nelson Textbook, Pages 198-204
Homework: Nelson Textbook: Page 205 #2, 3, 5, 8iab, 9, 10, 11, 12