4.3 Vertical and Horizontal Asymptotes

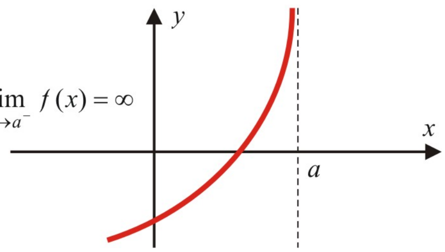
A Vertical Asymptote

If the value of $f(x)$ can be made *arbitrarily large* by taking x *sufficiently close* to a with $x<a$ then:

$$\lim_{x\to a^-} f(x)=\infty$$

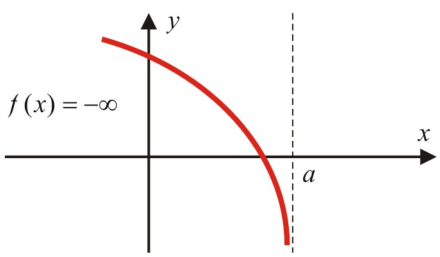
The line $x=a$ is called *vertical asymptote* to the graph of $y=f(x)$.

$$\lim_{x\to a^-} f(x)=\infty$$



Similarly:

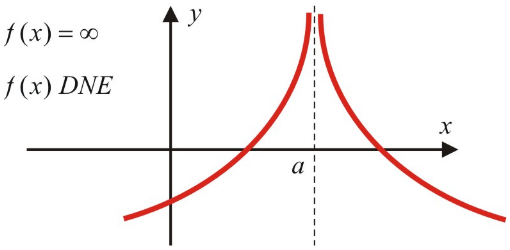
$$\lim_{x\to a^-} f(x)=-\infty$$



B Notes

$$\lim_{x\to a} f(x)=\infty$$

$$\lim_{x\to a} f(x)\ DNE$$

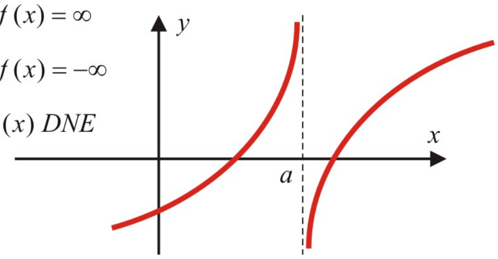


In this case, writing $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=\lim_{x\to a} f(x)=\infty$ is better than writing $\lim_{x\to a} f(x)\ DNE$.

$$\lim_{x\to a^-} f(x)=\infty$$

$$\lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to a} f(x)\ DNE$$



In this case, $\lim_{x\to a} f(x)\ DNE$.

C Rational Functions

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a vertical asymptote $x=a$ if:

$q(a)=0\ and\ p(a)\neq 0$

Ex 1. Find the equation of the vertical asymptote(s) for $y=f(x)=\dfrac{x-2}{x^2-4}$.

$$f(x)=\frac{x-2}{x^2-4}=\frac{x-2}{(x-2)(x+2)}=\frac{1}{x+2},\quad x\neq -2$$

Therefore, $x=-2$ is the equation of the vertical asymptote.

Ex 2. Find the behavior of the function

$y=f(x)=\dfrac{-x}{x^2+2x-3}$ near the vertical asymptotes.

$$f(x)=\frac{-x}{x^2+2x-3}=\frac{-x}{(x+3)(x-1)}$$

Vertical asymptotes are $x=-3$ and $x=1$.

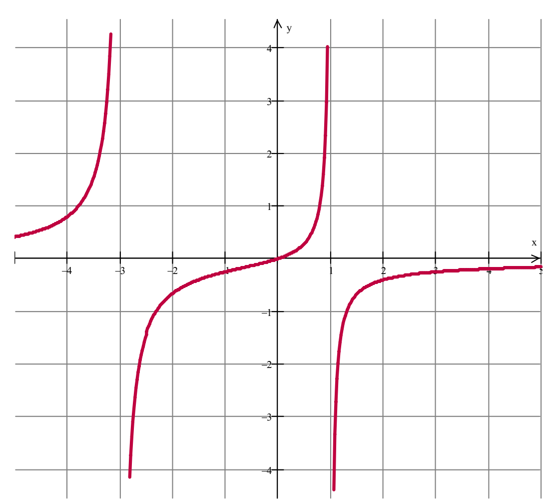
$$\lim_{x\to 1^+} f(x)=\frac{-1}{(1+3)(0^+)}=-\infty$$

$x=0$ is a simple zero of the numerator.

$x=-3$ and $x=1$ are simple zeros of the denominator.

The function $y=f(x)$ changes sign at each of these simple zeros.

The behavior of the function near the asymptotes is represented in the figure on the right.



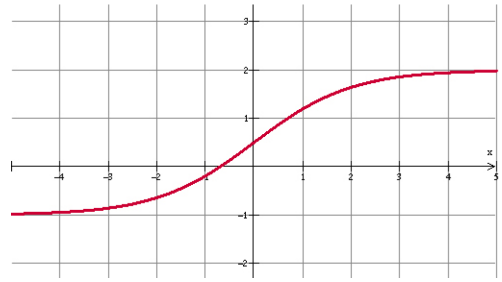
D Horizontal Asymptote

A horizontal line $y = L$ is called *horizontal asymptote* to the graph of $y = f(x)$ if:

$$\lim_{x\to\infty} f(x) = L \quad or \quad \lim_{x\to-\infty} f(x) = L$$

Notes.
1. A horizontal asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have *at most two* horizontal asymptotes (one as $x \to -\infty$ and one as $x \to \infty$) (see the figure on the right).

Ex 3. Find the equation of the horizontal asymptote(s) to the graph of the function $y = f(x)$ represented graphically below.



$\lim_{x\to-\infty} f(x) = -1, \quad \lim_{x\to\infty} f(x) = 2$

Therefore, $y = -1$ is the equation of the horizontal asymptote as $x \to -\infty$ and $y = 2$ is the equation of the horizontal asymptote as $x \to \infty$.

E Limits to Infinity

If $n \ge 1$, then:

$$\lim_{x\to\pm\infty} x^n = (\pm\infty)^n$$

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

Ex 4. Compute each limit.

a) $\lim_{x\to\infty} x = \infty$

b) $\lim_{x\to\infty} x^2 = \infty^2 = \infty$

c $\lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty$

d) $\lim_{x\to-\infty} x^4 = (-\infty)^4 = \infty$

F End behaviour

A polynomial functions

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$

behaves at infinity as the *leading term* $a_n x^n$.

Ex 5. Compute each limit.

a) $\lim_{x\to\infty}(-3x^4 + 5x^3 - 4)$

$\lim_{x\to\infty}(-3x^4 + 5x^3 - 4) = \lim_{x\to\infty}(-3x^4)$

$= (-3)\lim_{x\to\infty}(x^4) = (-3)(\infty^4) = (-3)(\infty) = -\infty$

b) $\lim_{x\to-\infty}(-x^3 - 2x^2 + x)$

$\lim_{x\to-\infty}(-x^3 - 2x^2 + x) = \lim_{x\to-\infty}(-x^3)$

$= (-1)\lim_{x\to-\infty} x^3 = (-1)(-\infty)^3 = (-1)(-\infty) = \infty$

G Rational Functions

To compute *limits at infinity* for a rational function, use the *end behavior* of the numerator and denominator:

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_2 x^2 + b_1 x + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$$

Ex 6. Compute each limit.

a) $\lim_{x\to\infty} \frac{-2x^3 + x}{5x^3 + x^2 + 1}$

$\lim_{x\to\infty} \frac{-2x^3 + x}{5x^3 + x^2 + 1} = \lim_{x\to\infty} \frac{-2x^3}{5x^3} = \lim_{x\to\infty} \frac{-2}{5} = \frac{-2}{5}$

b) $\lim_{x\to-\infty} \frac{3x^2 + 1}{-2x^3 + 3x}$

$\lim_{x\to-\infty} \frac{3x^2 + 1}{-2x^3 + 3x} = \lim_{x\to-\infty} \frac{3x^2}{-2x^3} = \frac{3}{-2} \lim_{x\to-\infty} \frac{1}{x} = \frac{3}{-2}(0) = 0$

c) $\lim_{x\to-\infty} \frac{3x^4 + x^2 - x}{-x^2 + x - 1}$

$\lim_{x\to-\infty} \frac{3x^4 + x^2 - x}{-x^2 + x - 1} = \lim_{x\to-\infty} \frac{3x^4}{-x^2} = (-3)\lim_{x\to-\infty} x^2$

$= (-3)(-\infty)^2 = (-3)(\infty) = -\infty$

H Horizontal Asymptotes for Rational Functions
A *rational function* of the form:

$$f(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+...+b_2x^2+b_1x+b_0}$$

has

- a horizontal asymptote $y=0$ if $m>n$
- a horizontal asymptote $y=\frac{a_n}{b_m}$ if $m=n$
- no horizontal asymptote if $n>m$

Note: A rational function may have *at most one* horizontal asymptote.

Ex 7. Find the equation of the horizontal asymptote.

a) $f(x)=\frac{3x^4+1}{-2x^4+3x^2}$

$m=n=4 \quad \Rightarrow \quad \therefore HA: y=-3/2$

b) $f(x)=\frac{-x^2+2x}{x^3-x^2+2}$

$n=2<m=3 \quad \Rightarrow \quad \therefore HA: y=0$

c) $f(x)=\frac{2x^2-3}{x+1}$

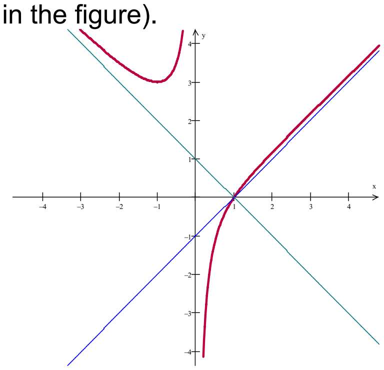
$n=2>m=1 \quad \Rightarrow \quad \therefore HA: none$

I Oblique (Slant) Asymptote
The line $y=ax+b$ is an *oblique (slant) asymptote* for the curve $y=f(x)$ if:

$$\lim_{x\to\pm\infty}[f(x)-(ax+b)]=0$$

Notes:
1. An oblique asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two oblique asymptotes (one as $x\to-\infty$ and one as $x\to\infty$).

Ex 8. Find the equations of the oblique asymptotes for the function represented below (oblique asymptotes are also represented in the figure).



$\therefore y=1-x$ (as $x\to-\infty$)
$\therefore y=-1+x$ (as $x\to\infty$)

J Oblique Asymptotes for Rational Functions
A rational function of the form:

$$f(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+...+b_2x^2+b_1x+b_0}$$

has an *oblique (slant) asymptote* if $n=m+1$.

Note. To get the equation of the oblique (slant) asymptote, use the *long division algorithm* to write the rational function in the form:

$$f(x)=\frac{P_n(x)}{Q_m(x)}=ax+b+\frac{R(x)}{Q_m(x)}$$

where $0\le degree(R)<degree(Q_m)=m$
Finally, the equation of the oblique (slant) asymptote is given by:

$$y=ax+b$$

Ex 9. Consider the rational function $y=f(x)=\frac{x^2}{x-1}$.

a) Find the equation of the oblique asymptote.

$y=f(x)=\frac{x^2}{x-1}=x+1+\frac{1}{x-1}$

Therefore, the equation of the oblique asymptote is $y=x+1$.

b) Find the derivative function and simplify.

$f'(x)=\frac{(2x)(x-1)-x^2(1)}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}=\frac{x(x-2)}{(x-1)^2}$

c) Find the local extrema.

$f'(x)=0 \ at\ x=0\ or\ x=2$

$f(0)=0, \quad f(2)=4$

x		0		1		2	
$f(x)$	↗	0	↘	DNE	↘	4	↗
$f'(x)$	+	0	-	DNE	-	0	+

At $(0,0)$ there is a local maximum point.
At $(2,4)$ there is a local minimum point.

Reading: Nelson Textbook, Pages 181-192
Homework: Nelson Textbook: Page 193 #1ab, 3bd, 4de, 5cd, 7ac, 9bd, 14