4.3 Vertical and Horizontal Asymptotes

A Vertical Asymptote
If the value of \( f(x) \) can be made arbitrarily large by taking \( x \) sufficiently close to \( a \) with \( x < a \) then:
\[
\lim_{x \to a^-} f(x) = \infty
\]
The line \( x = a \) is called a vertical asymptote to the graph of \( y = f(x) \).

Similarly:
\[
\lim_{x \to a^+} f(x) = -\infty
\]

B Notes
In this case, writing \( \lim_{x \to a} f(x) = \infty \) is better than writing \( \lim_{x \to a} f(x) = \infty \) is better than writing \( \lim_{x \to a} f(x) = \infty \).

C Rational Functions
A rational function of the form \( f(x) = \frac{p(x)}{q(x)} \) has a vertical asymptote \( x = a \) if:
\( q(a) = 0 \) and \( p(a) \neq 0 \)

Ex 1. Find the equation of the vertical asymptote(s) for \( y = f(x) = \frac{x-2}{x^2-4} \).

\[
f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}, \quad x \neq -2
\]
Therefore, \( x = -2 \) is the equation of the vertical asymptote.

Ex 2. Find the behavior of the function \( y = f(x) = \frac{-x}{x^2+2x-3} \) near the vertical asymptotes.

\[
f(x) = \frac{-x}{x^2+2x-3} = \frac{-x}{(x+3)(x-1)}
\]
Vertical asymptotes are \( x = -3 \) and \( x = 1 \).

\[
\lim_{x \to -3^+} f(x) = -\infty
\]
\( x = 0 \) is a simple zero of the numerator.
\( x = -3 \) and \( x = 1 \) are simple zeros of the denominator.
The function \( y = f(x) \) changes sign at each of these simple zeros.
The behavior of the function near the asymptotes is represented in the figure on the right.
**D Horizontal Asymptote**

A horizontal line \( y = L \) is called horizontal asymptote to the graph of \( y = f(x) \) if:

\[
\lim_{{x \to \pm \infty}} f(x) = L \quad \text{or} \quad \lim_{{x \to \pm \infty}} f(x) = L
\]

Notes:
1. A horizontal asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two horizontal asymptotes (one as \( x \to -\infty \) and one as \( x \to \infty \)) (see the figure on the right).

**Ex 3.** Find the equation of the horizontal asymptote(s) to the graph of the function \( y = f(x) \) represented graphically below.

\[
\lim_{{x \to -\infty}} f(x) = -1, \quad \lim_{{x \to \infty}} f(x) = 2
\]

Therefore, \( y = -1 \) is the equation of the horizontal asymptote as \( x \to -\infty \) and \( y = 2 \) is the equation of the horizontal asymptote as \( x \to \infty \).

**E Limits to Infinity**

If \( n \geq 1 \), then:

\[
\lim_{{x \to \pm \infty}} x^n = (\pm \infty)^n
\]

\[
\lim_{{x \to \pm \infty}} \frac{1}{x^n} = 0
\]

**F End behaviour**

A polynomial function

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0
\]

behaves at infinity as the leading term \( a_n x^n \).

**Ex 4.** Compute each limit.

a) \( \lim_{{x \to \infty}} x^2 = \infty^2 = \infty \)

b) \( \lim_{{x \to \infty}} x^3 = (-\infty)^3 = -\infty \)

c) \( \lim_{{x \to -\infty}} x^4 = (-\infty)^4 = \infty \)

**G Rational Functions**

To compute *limits at infinity* for a rational function, use the end behavior of the numerator and denominator:

\[
\lim_{{x \to \pm \infty}} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_2 x^2 + b_1 x + b_0} = \lim_{{x \to \pm \infty}} \frac{a_n}{b_m x^{m-n}}
\]

**Ex 5.** Compute each limit.

a) \( \lim_{{x \to \infty}} \frac{-2x^3 + x}{5x^3 + x^2 + 1} = \frac{-2}{5} \)

b) \( \lim_{{x \to \infty}} \frac{-2x^3+x}{5x^3+x^2+1} = \lim_{{x \to \infty}} -\frac{2x^3}{5x^3} = \lim_{{x \to \infty}} -\frac{2}{5} = -\frac{2}{5} \)

b) \( \lim_{{x \to -\infty}} \frac{3x^2+1}{-2x^3+3x} = \lim_{{x \to \infty}} \frac{3x^2}{-2x^3} = \lim_{{x \to \infty}} \frac{3}{-2} x = \frac{3}{-2} (0) = 0 \)

c) \( \lim_{{x \to \infty}} \frac{3x^4+x^2-x}{-x^2+x-1} = \lim_{{x \to \infty}} \frac{3x^4}{-x^2} = \lim_{{x \to \infty}} -3x^2 = (-3) \lim_{{x \to \infty}} x^2 = (-3)(\infty) = -\infty \)
H Horizontal Asymptotes for Rational Functions

A rational function of the form:
\[ f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0} \]
has
- a horizontal asymptote \( y = 0 \) if \( m > n \)
- a horizontal asymptote \( y = \frac{a_n}{b_m} \) if \( m = n \)
- no horizontal asymptote if \( n > m \)

Note: A rational function may have at most one horizontal asymptote.

Ex 7. Find the equation of the horizontal asymptote.

a) \( f(x) = \frac{3x^4 + 1}{-2x^4 + 3x^2} \)
   \[ m = n = 4 \implies \text{HA: } y = -\frac{3}{2} \]

b) \( f(x) = \frac{-x^2 + 2x}{x^3 - x^2 + 2} \)
   \[ n = 2 < m = 3 \implies \text{HA: } y = 0 \]

c) \( f(x) = \frac{2x^2 - 3}{x + 1} \)
   \[ n = 2 > m = 1 \implies \text{HA: none} \]

I Oblique (Slant) Asymptote

The line \( y = ax + b \) is an oblique (slant) asymptote for the curve \( y = f(x) \) if:
\[ \lim_{x \to \pm\infty} \left( f(x) - (ax + b) \right) = 0 \]

Notes:
1. An oblique asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two oblique asymptotes (one as \( x \to -\infty \) and one as \( x \to \infty \)).

J Oblique Asymptotes for Rational Functions

A rational function of the form:
\[ f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0} \]
has an oblique (slant) asymptote if \( n = m + 1 \).

Note. To get the equation of the oblique (slant) asymptote, use the long division algorithm to write the rational function in the form:
\[ f(x) = \frac{P_n(x)}{Q_m(x)} = ax + b + \frac{R(x)}{Q_m(x)} \]
where \( 0 \leq \text{degree}(R) < \text{degree}(Q_m) = m \)

Finally, the equation of the oblique (slant) asymptote is given by:
\[ y = ax + b \]

Ex 8. Find the equations of the oblique asymptotes for the function represented below (oblique asymptotes are also represented in the figure).

\[ \therefore y = 1 - x \quad \text{(as } x \to -\infty \text{)} \]
\[ \therefore y = -1 + x \quad \text{(as } x \to \infty \text{)} \]

Ex 9. Consider the rational function \( y = f(x) = \frac{x^2}{x-1} \).

a) Find the equation of the oblique asymptote.
\[ y = f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1} \]
Therefore, the equation of the oblique asymptote is
\[ y = x + 1 \]

b) Find the derivative function and simplify.
\[ f'(x) = \frac{(2x)(x-1) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2} \]

\[ f'(x) = 0 \text{ at } x = 0 \text{ or } x = 2 \]
\[ f(0) = 0, \quad f(2) = 4 \]

\[
\begin{array}{c|c|c|c|c}
  x & 0 & 1 & 2 \\
  f(x) & \nearrow & 0 & \dne & \searrow & 4 & \nearrow \\
  f'(x) & + & 0 & - & \dne & - & 0 & + \\
\end{array}
\]

At \((0,0)\) there is a local maximum point.
At \((2,4)\) there is a local minimum point.

Reading: Nelson Textbook, Pages 181-192
Homework: Nelson Textbook: Page 193 #1ab, 3bd, 4de, 5cd, 7ac, 9bd, 14