4.2 Critical Points. Local Maxima and Minima

**A Local Maximum**
A function \( f \) has a **local (relative) maximum** at \( x = c \) if \( f(x) \leq f(c) \) when \( x \) is sufficiently close to \( c \) (from both sides).

\( f(c) \) is called the **local (relative) maximum value**.

\( (c, f(c)) \) is called the **local (relative) maximum point**.

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**B Local Minimum**
A function \( f \) has a **local (relative) minimum** at \( x = c \) if \( f(x) \geq f(c) \) when \( x \) is sufficiently close to \( c \) (from both sides).

\( f(c) \) is called the **local (relative) minimum value**.

\( (c, f(c)) \) is called the **local (relative) minimum point**.

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**C Note**
The following points are neither local minimum or maximum points.

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**D Critical Numbers and Critical Points**
The number \( c \in D \) is a **critical number** if either 

\( f'(c) = 0 \) or \( f'(c) \text{ DNE} \)

The point \( P(c, f(c)) \) is called a **critical point**.

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**Ex 1.** A function is defined by the following graph. Find the local minimum or maximum points.

Local minimum points are \( A(-5, 2) \), \( B(-1, 1) \), and \( C(5, 2) \).

Local maximum points are \( D(-6, 4) \) and \( E(2, 5) \).

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**Ex 2.** Find the critical points for:

a) \( f(x) = 2x^3 + 3x^2 \)

\[ f'(x) = 6x^2 + 6x = 6x(x + 1) \]

\[ f'(x) = 0 \Rightarrow x = 0 \text{ or } x = -1 \]

\[ f(0) = 0 \quad f(-1) = 2(-1)^3 + 3(-1)^2 = -2 + 3 = 1 \]

The critical points are \( A(0, 0) \) and \( B(-1, 1) \).

b) \( f(x) = \sqrt[3]{x^2 - 1} \)

\[ f'(x) = \left[x^2 - 1\right]^{1/3} = \frac{1}{3}(x^2 - 1)^{-2/3}(2x) = \frac{2x}{3\sqrt[3]{(x^2 - 1)^2}} \]

\[ f'(x) = 0 \Rightarrow x = 0, y = -1 \]

\[ f'(x) \text{ DNE} \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1, y = 0 \]

The critical points are \( A(0, -1) \), \( B(-1, 0) \), and \( C(1, 0) \).
E Fermat's Theorem
If \( f \) has a local extremum (minimum or maximum) at \( x = c \) then \( c \) is a critical number for \( f \). So, at a local extremum:
\[
f'(c) = 0 \text{ or } f'(c) \text{ DNE}
\]

Note.
If
\[
f'(c) = 0 \text{ or } f'(c) \text{ DNE}
\]
then the function \( f \) may have or not a local extremum (minimum or maximum) at \( x = c \).

F First Derivative Test
Let \( c \) be a critical number of a continuous function \( f \).
If \( f'(x) \) changes sign from positive to negative at \( c \) then \( f \) has a local maximum at \( c \).
If \( f'(x) \) changes sign from negative to positive at \( c \) then \( f \) has a local minimum at \( c \).
If \( f'(x) \) does not change sign at \( c \) then \( f \) has no local extremum (minimum or maximum) at \( c \).

Ex 3. Find the local extrema points for the functions in the previous example.

a) \( f(x) = 2x^3 + 3x^2 \)

\[
f'(x) = 6x^2 + 6x = 6x(x+1)
\]

Sign Chart:

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>(0)</th>
<th>(+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(\nearrow)</td>
<td>(\downarrow)</td>
<td>(\uparrow)</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>(+)</td>
<td>(\text{DNE})</td>
<td>(-)</td>
</tr>
</tbody>
</table>

The point \( A(0,0) \) is a local minimum point and the point \( B(-1,1) \) is a local maximum point.

b) \( f(x) = \sqrt{x^2 - 1} \)

\[
f'(x) = \frac{2x}{3(x^2 - 1)^{3/2}}
\]

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</tr>
</tbody>
</table>

The point \( A(0,-1) \) is a local minimum point.

Ex 4. Find a function of the form \( f(x) = ax^4 + bx^2 + cx + d \) with a local maximum at \((0,-6)\) and a local minimum at \((1,-8)\).

\[
f(0) = -6 \Rightarrow d = -6 \quad (1)
\]
\[
f(1) = -8 \Rightarrow a + b + c + d = -8 \quad (2)
\]
\[
f'(x) = 4ax^3 + 2bx + c
\]
\[
f'(0) = 0 \Rightarrow c = 0 \quad (3)
\]
\[
f'(1) = 0 \Rightarrow 4a + 2b + c = 0 \quad (4)
\]

\[(1),(2),(3),(4) \Rightarrow \begin{cases} a + b - 6 = -8 \\ 4a + 2b = 0 \end{cases} \Rightarrow \begin{cases} a + b = -2 \\ 2a + b = 0 \end{cases} \Rightarrow a = 2 \quad \text{and} \quad b = -4
\]

\[
\therefore f(x) = 2x^4 - 4x^2 - 6
\]