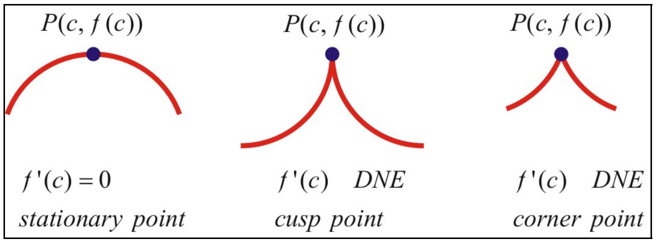
4.2 Critical Points. Local Maxima and Minima

A Local Maximum

A function f has a *local (relative) maximum* at $x = c$ if $f(x) \le f(c)$ when x is sufficiently close to c (from both sides).

$f(c)$ is called the local (relative) maximum *value*.

$(c, f(c))$ is called the local (relative) maximum *point*.

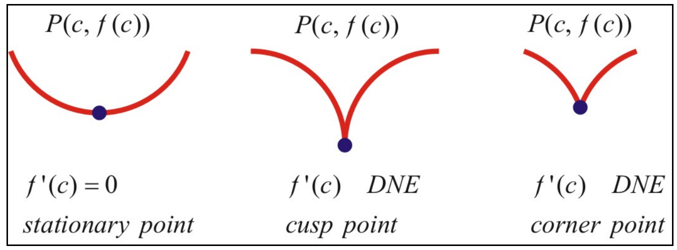


B Local Minimum

A function f has a *local (relative) minimum* at $x = c$ if $f(x) \ge f(c)$ when x is sufficiently close to c (from both sides).

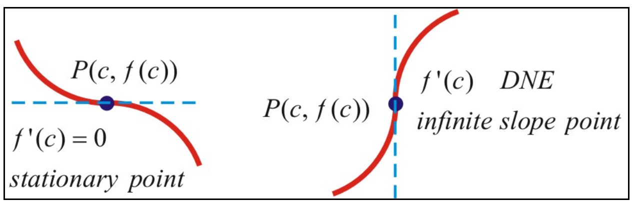
$f(c)$ is called the local (relative) minimum *value*.

$(c, f(c))$ is called the local (relative) minimum *point*.

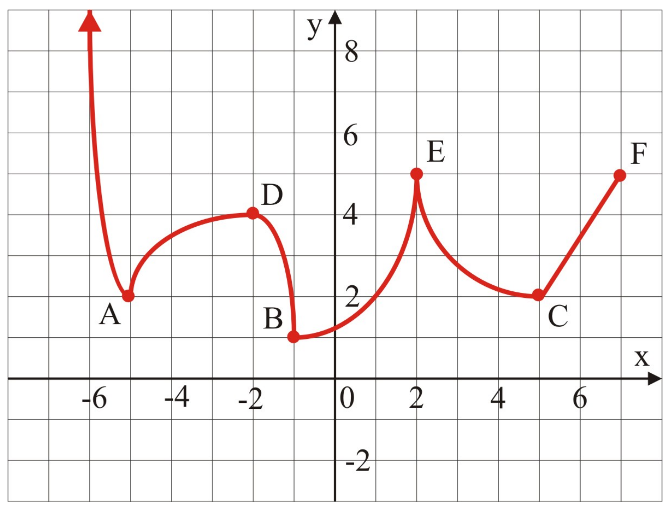


C Note

The following points are neither local minimum or maximum points.



Ex 1. A function is defined by the following graph. Find the local minimum or maximum points.



Local minimum points are $A(-5,2)$, $B(-1,1)$, and $C(5,2)$.

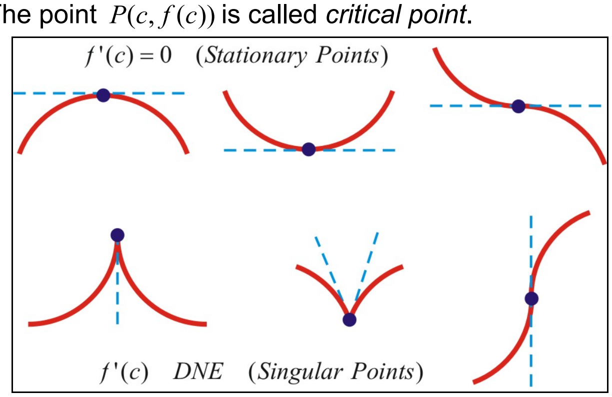
Local maximum points are $D(-2,4)$ and $E(2,5)$.

D Critical Numbers and Critical Points

The number $c \in D_f$ is a *critical number* if either

$$f'(c) = 0 \text{ or } f'(c) \; DNE$$

The point $P(c, f(c))$ is called *critical point*.



Ex 2. Find the critical points for:

a) $f(x) = 2x^3 + 3x^2$

$f'(x) = 6x^2 + 6x = 6x(x+1)$

$f'(x) = 0 \Rightarrow x = 0 \text{ or } x = -1$

$f(0) = 0 \quad f(-1) = 2(-1)^3 + 3(-1)^2 = -2 + 3 = 1$

The critical points are $A(0,0)$ and $B(-1,1)$.

b) $f(x) = \sqrt[3]{x^2 - 1}$

$$f'(x) = \left[(x^2-1)^{1/3}\right]' = \frac{1}{3}(x^2-1)^{-2/3}(2x) = \frac{2x}{3\sqrt[3]{(x^2-1)^2}}$$

$f'(x) = 0 \Rightarrow x = 0, y = -1$

$f'(x) \; DNE \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1, y = 0$

The critical points are $A(0,-1)$, $B(-1,0)$, and $C(1,0)$.

E Fermat's Theorem

If f has a local extremum (minimum or maximum) at $x = c$ then c is a critical number for f. So, at a local extremum:

$$f'(c) = 0 \text{ or } f'(c)\ DNE$$

Note.

If

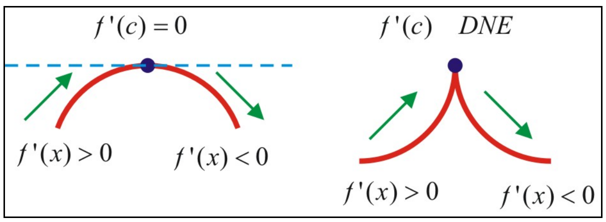
$$f'(c) = 0 \text{ or } f'(c)\ DNE$$

then the function f may have or not a local extremum (minimum or maximum) at $x = c$.

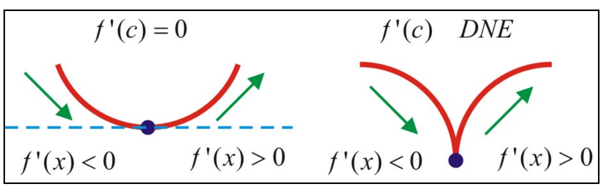
F First Derivative Test

Let c be a critical number of a *continuous* function f.

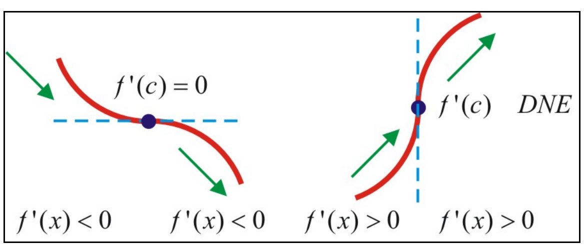
If $f'(x)$ changes sign from positive to negative at c then f has a *local maximum* at c.



If $f'(x)$ changes sign from negative to positive at c then f has a *local minimum* at c.



If $f'(x)$ does not change sign at c then f has no local extremum (minimum or maximum) at c.



Ex 3. Find the local extrema points for the functions in the previous example.

a) $f(x) = 2x^3 + 3x^2$

$f'(x) = 6x^2 + 6x = 6x(x+1)$

Sign Chart:

x		-1		0	
$f(x)$	↗	1	↘	0	↗
$f'(x)$	+	0	-	0	+

The point $A(0,0)$ is a local minimum point and the point $B(-1,1)$ is a local maximum point.

b) $f(x) = \sqrt[3]{x^2 - 1}$

$$f'(x) = \frac{2x}{3\sqrt[3]{(x^2-1)^2}}$$

Sign Chart:

x		-1		0		1	
$f(x)$	↘	0	↘	-1	↗	0	↗
$f'(x)$	-	DNE	-	0	+	DNE	+

The point $A(0,-1)$ is a local minimum point.

Ex 4. Find a function of the form $f(x) = ax^4 + bx^2 + cx + d$ with a local maximum at $(0,-6)$ and a local minimum at $(1,-8)$.

$f(0) = -6 \Rightarrow d = -6 \quad (1)$

$f(1) = -8 \Rightarrow a + b + c + d = -8 \quad (2)$

$f'(x) = 4ax^3 + 2bx + c$

$f'(0) = 0 \Rightarrow c = 0 \quad (3)$

$f'(1) = 0 \Rightarrow 4a + 2b + c = 0 \quad (4)$

$$(1),(2),(3),(4) \Rightarrow \begin{cases} a + b + 0 - 6 = -8 \\ 4a + 2b + 0 = 0 \end{cases} \Rightarrow \begin{cases} a + b = -2 \\ 2a + b = 0 \end{cases} \Rightarrow$$

$a = 2 \quad and \quad b = -4$

$\therefore f(x) = 2x^4 - 4x^2 - 6$

Reading: Nelson Textbook, Pages 172-178

Homework: Nelson Textbook: Page 178 #3ab, 5ad, 7adf, 10, 13, 15