3.3 3.4 Optimization

A Algorithm for Solving Optimization Problems

1. Read and understand the problem's text.
2. Draw a diagram (if necessary).
3. Assign variables to the quantities involved and state restrictions according to the situation.
4. Write relations between these variables.
5. Identify the variable that is minimized or maximized. This is the dependant variable.
6. Use the other relations (called constraints) to express the dependent variable (the one which is minimized or maximized) as a function of one single variable (the independent variable).
7. Find extrema (maximum or minimum) for the dependant variable (using global extrema algorithm, first derivative test or the second derivative test).
8. Check if extrema satisfy the conditions of the application.
9. Find the value of other variables at extrema (if necessary).
10. Write the conclusion statement.

B Optimization Problems Involving Numbers

Ex 1. Find two positive numbers with a product equal to 200 such that the sum of one number and twice the other number is as small as possible. What is the minimum value of the sum?

Let x and y be the two numbers.

$x, y \in R, \quad x, y > 0$ (restrictions)

$xy = 200$ (constraint)

Let s be the sum of first number x and twice the other number y:

$s = x + 2y$

The task is to minimize s.

$y = \frac{200}{x}$

$s = x + 2\frac{200}{x} = x + \frac{400}{x}$

$s' = 1 - \frac{400}{x^2}$

$s' = 0 \; when \; 1 - \frac{400}{x^2} = 0 \Rightarrow x^2 = 400 \Rightarrow x = \pm 20$

Because $x > 0$, $x = 20$ and $y = \frac{200}{20} = 10$

$s'' = \frac{800}{x^3} > 0$ for $x > 0$

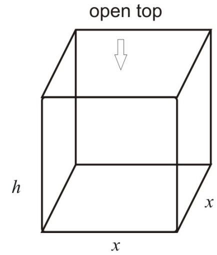
So s has a minimum value when $s' = 0$. The minimum value of the s is:

$s - \min = 20 + 2(10) = 40$

$\therefore$ The minimum value of the sum is 40. This minimum is achieved when the first number is 20 and the second number is 10.

C Maximize the Volume given the Shape and Area

Ex 2. If $2700cm^2$ of material is available to make a box with *a square base and an open top*, find the dimensions (length, width, and height) of the box that give the largest volume of the box. What is the maximum volume of the box?



Let x be the side length of the base, h be the height, and V be the volume of the box.

$x, h, V \in R, \quad x, h, V > 0$ (restrictions)

The total lateral area is:

$x^2 + 4xh = 2700$ (constraint)

The volume $V = x^2 h$ is to be maximized.

So:

$$h = \frac{2700 - x^2}{4x}$$

$$V = x^2 \frac{2700 - x^2}{4x} = \frac{2700x - x^3}{4}$$

$$V' = \frac{2700 - 3x^2}{4}$$

$$V' = 0 \; when \; 2700 - 3x^2 = 0 \Rightarrow x = \pm 30$$

Because $x > 0$, $x = 30cm$, and $h = \frac{2700 - 30^2}{4(30)} = 15cm$.

$V'' = \frac{-6x}{4} < 0$ when $x > 0$.

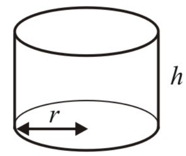
So, the volume is maximum when $V' = 0$. The maximum volume is:

$V - \max = 30^2(15) = 13500cm^3$

$\therefore$ The maximum value of the volume is $13500cm^3$. This maximum is achieved when the base side is $30cm$ and the height of the box is $15cm$.

D Minimize the Area given the Shape and Volume

Ex 3. A cylindrical can is to be made to hold $1000cm^3$ (one litre) of oil. Find the dimensions (radius and height) of the can that will minimize the cost of the metal to make the can.



The minimum cost corresponds to a minimum of the total area of the can (cylinder).

Let r be the radius, h be the height and A be the total area of the can.

$r, h, A \in R, \quad r, h, A > 0$ (restrictions)

The volume of the can is given by:

$1000 = \pi r^2 h$ (constraint)

The total area of the cylinder $A = 2\pi r^2 + 2\pi r h$ is to be minimized.

So:

$$h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

$$A' = 4\pi r - \frac{2000}{r^2}$$

$$A' = 0 \; when \; 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow$$

$$r = \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}} \cong 5.42cm$$

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} \cong 10.84cm$$

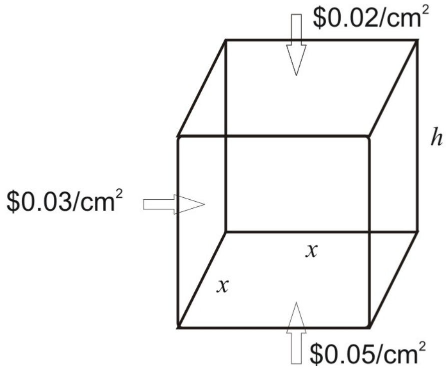
$$A'' = 4\pi + \frac{4000}{r^3} > 0 \quad because \quad r > 0$$

The area is minimum when $A' = 0$.

$\therefore$ The can has a minimum cost when the radius is $5.42cm$ and the height is $10.84cm$.

E Minimize the Cost given the Shape and Volume

Ex 4. A closed box with a square base is to contain $252cm^3$. The bottom costs $\$0.05/cm^2$, the top costs $\$0.02/cm^2$ and the sides costs $\$0.03/cm^2$. Find the dimensions (base side and height) that will minimize the cost.



Let x be the side length of the base and h be the height.

$x, h \in R, \quad x, h > 0$ (restrictions)

The volume is given by:

$252 = x^2 h$ (constraint)

The cost C of the box that must be minimized is given by:

$C = (0.05)x^2 + (0.03)4xh + (0.02)x^2$

So:

$$h = \frac{252}{x^2}$$

$$C = 0.07x^2 + 0.12x\frac{252}{x^2} = 0.07x^2 + \frac{30.24}{x}$$

$$C' = 0.14x - \frac{30.24}{x^2}$$

$$C' = 0 \ when \ 0.14x - \frac{30.24}{x^2} = 0 \Rightarrow x = \sqrt[3]{\frac{30.24}{0.14}} = 6$$

$$h = \frac{252}{6^2} = 7$$

$$C'' = 0.14 + \frac{60.48}{x^3} > 0 \ beacuse \ x > 0$$

The minimum cost is:

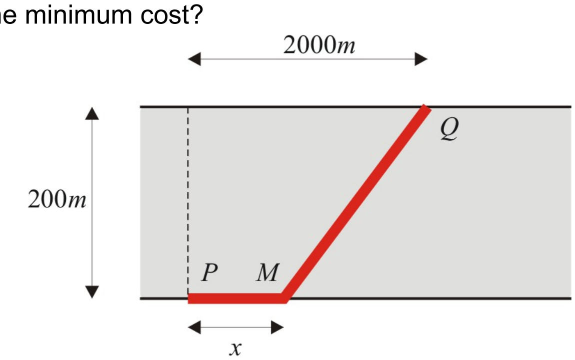
$$C - \min = (0.07)6^2 + \frac{30.24}{6} = \$7.56$$

$\therefore$ The minimum cost is $\$7.56$ when the base side is $6cm$ and the height is $7cm$.

F Minimize the Cost for an Underground/Underwater Cable

Ex 5. A cable television company is laying cable in an area with underground utilities. Two subdivisions are located on opposite sides of a $200m$ wide river. The company has to connect points P and Q with cable, where P is on the South bank and Q is on the North bank $2000m$ East of P. It cost $\$8/m$ to lay cable underground and $\$10/m$ to lay cable underwater.

What is the least expensive way to lay the cable and what is the minimum cost?



Let x be the distance the cable is laid underground. Then:

$0 \le x \le 2000m$ (restriction)

The cost is given by:

$C = 8x + 10\sqrt{200^2 + (2000 - x)^2}$

So:

$$C' = 8 + 10\frac{2(2000 - x)(-1)}{2\sqrt{200^2 + (2000 - x)^2}}$$

$$= 8 - 10\frac{2000 - x}{\sqrt{200^2 + (2000 - x)^2}}$$

$$C' = 0 \ when \ 8 = 10\frac{2000 - x}{\sqrt{200^2 + (2000 - x)^2}} \Rightarrow$$

$$0.8^2[200^2 + (2000 - x)^2] = (2000 - x)^2$$

$$25600 = 0.36(2000 - x)^2$$

$$2000 - x = \pm\sqrt{25600/0.36}$$

$$x = 2000 \pm \sqrt{25600/0.36} = 2000 \pm 266.67$$

$$x = 2000 - 266.67 \cong 1733.33m \ because \ x \le 2000m$$

Let use the global extrema algorithm to find the minimum cost of the function:

$C = 8x + 10\sqrt{200^2 + (2000 - x)^2}$ over $[0, 2000]$.

$$C(0) = 10\sqrt{200^2 + 2000^2} \cong \$20{,}997.51$$

$$C(2000) = 8(2000) + 10(200) = \$18{,}000$$

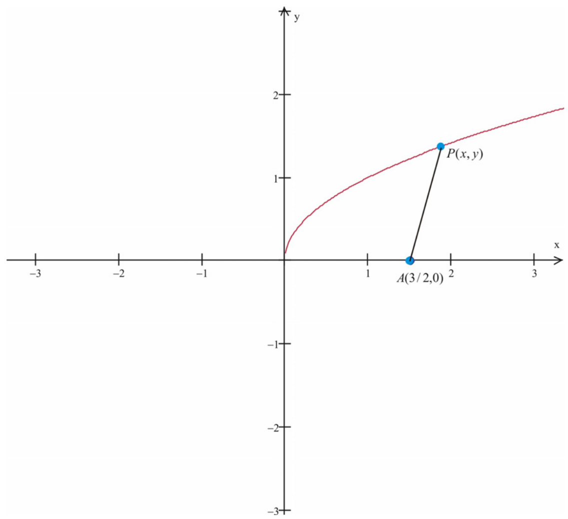
$$C(1912.72) = 8(1733.33) + 10\sqrt{200^2 + (2000 - 1733.33)^2}$$

$$= \$17{,}200$$

$\therefore$ The minimum cost is $\$17{,}200$ is achieved when the cable is laid underground over $1733.33m$.

G Geometry

Ex 6. How close does the curve $y=\sqrt{x}$ come to the point $(3/2,0)$?



Let $P(x,y)$ be a point on the curve $y=\sqrt{x}$.

$x,y\in R,\quad x,y\ge 0$ (restrictions)

The distance $D=AP=\sqrt{(x-3/2)^2+(y-0)^2}$ is to be minimized.

But: $y=\sqrt{x}$ (constraint).

Is preferable to minimize D^2:

$D^2=(x-3/2)^2+(y-0)^2=(x-3/2)^2+x$

$\frac{d}{dx}D^2=2(x-3/2)+1$

$\frac{d}{dx}D^2=0\ when\ 2(x-3/2)+1=0\Rightarrow x=1, y=1$

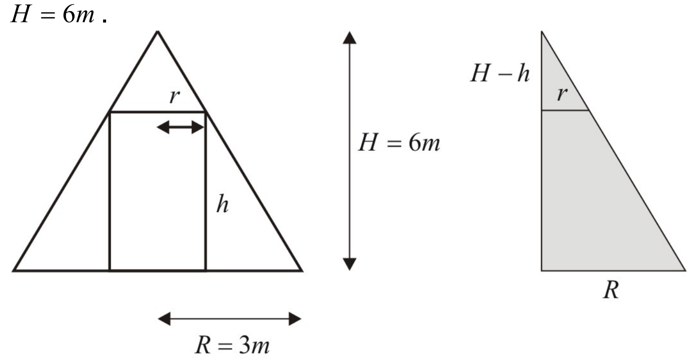
$\frac{d^2}{dx^2}D^2=2>0$

So, the minimum distance from the point $(3/2,0)$ to the curve $y=\sqrt{x}$ happens when $x=1$ and $y=1$. This minimum distance is:

$D-\min\sqrt{(1-3/2)^2+(1-0)^2}=\sqrt{1/4+1}=\sqrt{5}/2$

$\therefore$ The minimum distance from the point $(3/2,0)$ to the curve $y=\sqrt{x}$ is $\sqrt{5}/2$.

Ex 7. Find the dimensions of the largest right-cylinder that can be inscribed in a cone of radius $R=3m$ and height $H=6m$.



Let r be the radius and h be the height of the right-cylinder.

$r,h\in R,\quad r,h>0$

By using similar triangles:

$\frac{H-h}{r}=\frac{H}{R}\Rightarrow H-h=\frac{rH}{R}\Rightarrow h=H\left(1-\frac{r}{R}\right)$ (constraint)

The volume of the right-cylinder $V=\pi r^2h$ must be maximized.

So:

$V=\pi r^2H\left(1-\frac{r}{R}\right)=\pi r^2H-\frac{\pi r^3H}{R}$

$V'=2\pi rH-\frac{3\pi r^2H}{R}=\pi rH\left(2-\frac{3r}{R}\right)$

$V'=0\ at\ r=0\ or\ r=\frac{2R}{3}$

$V''=2\pi H-\frac{6\pi rH}{R}$

Because $r>0$ the extremum happens at $r=\frac{2R}{3}$ and $h=H\left(1-\frac{2}{3}\right)=\frac{H}{3}$. At this extremum

$V''=2\pi H-\frac{6\pi H}{R}\frac{2R}{3}=-2\pi H<0$

$V=\pi\left(\frac{2R}{3}\right)^2\frac{H}{3}=\frac{4\pi R^2H}{27}$

Numerical Case:

$r=\frac{2R}{3}=\frac{2(3)}{3}=2m$

$h=\frac{H}{3}=\frac{6}{3}=2m$

$V=\frac{4\pi R^2H}{27}=\frac{4\pi 3^2(6)}{27}=8\pi\ m^3$

$\therefore$ The right-cylinder has a minimum volume of $8\pi m^3$ when the radius is $2m$ and the height is $2m$.

Reading: Nelson Textbook, Pages 141-144 (Optimization Problems)
Homework: Nelson Textbook: Page 145 #1, 4, 7, 10, 16, 20
Reading: Nelson Textbook, Pages 148-150 (Optimization Problems in Economics and Science)
Homework: Nelson Textbook: Page 151 #5, 10, 12, 17