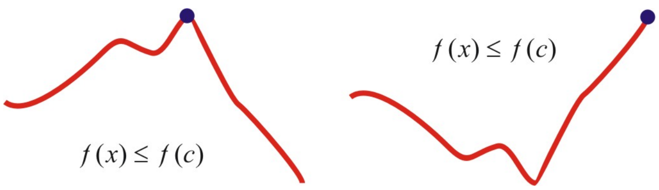
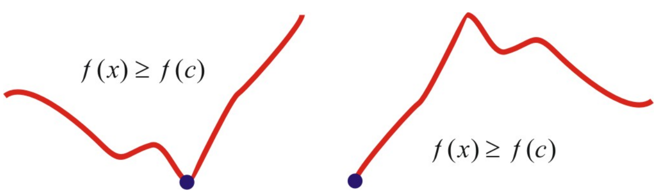
3.2 Maximum and Minimum on an Interval. Extreme Values

A Global Maximum
A function f has a *global (absolute) maximum* at $x = c$ if $f(x) \le f(c)$ for all $x \in D_f$.
$f(c)$ is called the *global (absolute) maximum value*.
$(c, f(c))$ is called the *global (absolute) maximum point*.

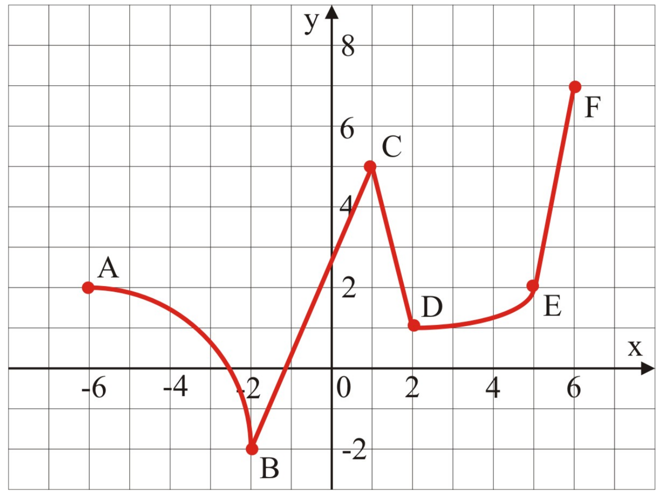


B Global Minimum
A function f has a *global (absolute) minimum* at $x = c$ if $f(x) \ge f(c)$ for all $x \in D_f$.
$f(c)$ is called the *global (absolute) minimum value*.
$(c, f(c))$ is called the *global (absolute) minimum point*.



C Extremum and Extrema
An *extremum* is either a minimum or a maximum (value, point, local or global).
Extrema is the plural of extremum.

Ex 1. Find extrema for the function represented in the figure below by its graph.



Local minimum points are $B(-2,-2)$ and $D(2,1)$.
Local maximum points is $C(1,5)$.
Global minimum point is $B(-2,-2)$.
Global maximum point is $F(6,7)$.

D Global (Absolute) Extrema Algorithm
To find the global (absolute) extrema for a *continuous* function f over a closed interval $[a,b]$:
1) identify the *critical* numbers over (a,b)
2) find the *values* of the function $f(c)$ at each critical number c in (a,b)
3) find the *values* $f(a)$ and $f(b)$
4) from the values obtained at part 2) and 3):
- the *largest* represents the *global (absolute) maximum* value
- the *least* represents the *global (absolute) minimum* value

Note. c is a critical number if either $f'(c) = 0$ or $f'(c)\ DNE$

Ex 2. Find extrema for $f(x) = 3x^4 - 4x^3$ over $[-1,2]$.

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$
$f'(x) = 0 \Rightarrow x = 0 \quad or \quad x = 1$
$x = 0$ and $x = 1$ are critical numbers in $(-1,2)$.
$f(0) = 0$
$f(1) = 3 - 4 = -1$
$f(-1) = 3(-1)^4 - 4(-1)^3 = 3 + 4 = 7$
$f(2) = 3(2)^4 - 4(2)^3 = 48 - 32 = 16$

Therefore:
- the global minimum point is $A(1,-1)$
- the global maximum point is $B(2,16)$

Ex 3. Find extrema for $f(x)=\frac{x^2-1}{x^2+1}$ over $[-1,3]$.

$$f'(x)=\frac{2x(x^2+1)-(x^2-1)(2x)}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

$f'(x)=0 \Rightarrow x=0$ (critical number in $(-1,3)$)

$f(0)=-1$

$f(-1)=0$

$$f(3)=\frac{3^2-1}{3^2+1}=\frac{8}{10}=0.8$$

Therefore:

- the global minimum point is $A(0,-1)$
- the global maximum point is $B(3,0.8)$

Ex 4. Find extrema for $f(x)=(x+1)^2\sqrt[3]{x-1}$ over $[0,1/2]$.

$$f'(x)=2(x+1)(x-1)^{\frac{1}{3}}+(x+1)^2\frac{1}{3}(x-1)^{-\frac{2}{3}}$$

$$=2(x+1)(x-1)^{\frac{1}{3}}\frac{3}{3}(x-1)^{-\frac{2}{3}}(x-1)^{\frac{2}{3}}+(x+1)^2\frac{1}{3}(x-1)^{-\frac{2}{3}}$$

$$=\frac{(x+1)[6(x-1)+(x+1)]}{3\sqrt[3]{(x-1)^2}}=\frac{(x+1)(7x-5)}{3\sqrt[3]{(x-1)^2}}$$

$f'(x)=0 \Rightarrow x=-1 \quad or \quad x=5/7$. These critical numbers are not in the interval $(0,1/2)$.

$f'(x)\ DNE$ at $x=1$. This critical number is not in the interval $(0,1/2)$).

$f(0)=(0+1)^2\sqrt[3]{0-1}=1(-1)=-1$

$$f(1/2)=(1/2+1)^2\sqrt[3]{1/2-1}=-\frac{9}{4\sqrt[3]{2}}\cong -1.786$$

Therefore:

- the global minimum point is $A(0.5,-1.786)$
- the global maximum point is $B(0,-1)$

Reading: Nelson Textbook, Pages 130-134
Homework: Nelson Textbook: Page 136 #3ace, 4abe, 7ab, 10, 11

