3.2 Maximum and Minimum on an Interval. Extreme Values

**A Global Maximum**
A function \( f \) has a **global (absolute) maximum** at \( x = c \) if \( f(x) \leq f(c) \) for all \( x \in D_f \).

\( f(c) \) is called the **global (absolute) maximum value**. 
\((c, f(c))\) is called the **global (absolute) maximum point**.

**B Global Minimum**
A function \( f \) has a **global (absolute) minimum** at \( x = c \) if \( f(x) \geq f(c) \) for all \( x \in D_f \).

\( f(c) \) is called the **global (absolute) minimum value**. 
\((c, f(c))\) is called the **global (absolute) minimum point**.

**C Extremum and Extrema**
An **extremum** is either a minimum or a maximum (value, point, local or global). 
**Extrema** is the plural of extremum.

**D Global (Absolute) Extrema Algorithm**
To find the global (absolute) extrema for a **continuous** function \( f \) over a closed interval \([a,b]\):
1) Identify the **critical numbers** over \((a,b)\)
2) Find the **values** of the function \( f(c) \) at each critical number \( c \) in \((a,b)\)
3) Find the **values** \( f(a) \) and \( f(b) \)
4) From the values obtained at part 2) and 3):
   - the **largest** represents the **global (absolute) maximum** value 
   - the **least** represents the **global (absolute) minimum** value

**Note.** \( c \) is a critical number if either \( f'(c) = 0 \) or \( f'(c) \ DNE \)

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**Ex 1.** Find extrema for the function represented in the figure below by its graph.

Local minimum points are \( B(-2,-2) \) and \( D(2,1) \).
Local maximum points is \( C(1,5) \).
Global minimum point is \( B(-2,-2) \).
Global maximum point is \( F(6,7) \).

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**Ex 2.** Find extrema for \( f(x) = 3x^4 - 4x^3 \) over \([-1,2]\).

\[ f'(x) = 12x^3 - 12x^2 = 12x^2(x-1) \]
\[ f'(x) = 0 \Rightarrow x = 0 \text{ or } x = 1 \]
\[ x = 0 \text{ and } x = 1 \text{ are critical numbers in } (-1,2) \]
\[ f(0) = 0 \]
\[ f(1) = 3 - 4 = -1 \]
\[ f(-1) = 3(-1)^4 - 4(-1)^3 = 3 + 4 = 7 \]
\[ f(2) = 3(2)^4 - 4(2)^3 = 48 - 32 = 16 \]

Therefore:
- the global minimum point is \( A(1,-1) \)
- the global maximum point is \( B(2,16) \)
Ex 3. Find extrema for \( f(x) = \frac{x^2 - 1}{x^2 + 1} \) over \([-1,3]\).

\[
f'(x) = \frac{2x(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}
\]

\( f'(x) = 0 \implies x = 0 \) (critical number in \((-1,3)\))

\[
f(0) = -1
\]

\[
f(-1) = 0
\]

\[
f(3) = \frac{3^2 - 1}{3^2 + 1} = \frac{8}{10} = 0.8
\]

Therefore:

- the global minimum point is \( A(0, -1) \)
- the global maximum point is \( B(3, 0.8) \)

Ex 4. Find extrema for \( f(x) = (x+1)^2 \sqrt[3]{x-1} \) over \([0,1/2]\).

\[
f'(x) = 2(x+1)(x-1)^{\frac{1}{3}} + (x+1)^2 \frac{1}{3}(x-1)^{-\frac{2}{3}}
\]

\[
= 2(x+1)(x-1)^{\frac{1}{3}} + 2 \frac{1}{3}(x-1)^{-\frac{2}{3}} + (x+1)^2 \frac{1}{3}(x-1)^{-\frac{2}{3}}
\]

\[
= x+1 [6(x-1) + (x+1)] = \frac{(x+1)(7x-5)}{3 \sqrt[3]{x-1}^2}
\]

\[
f'(x) = 0 \implies x = -1 \text{ or } x = 5/7 \). These critical numbers are not in the interval \((0,1/2)\).

\( f'(x) \ DNE \) at \( x = 1 \). This critical number is not in the interval \((0,1/2)\).

\[
f(0) = (0+1)^2 \sqrt[3]{0-1} = 1(-1) = -1
\]

\[
f(1/2) = (0.5+1)^2 \sqrt[3]{1/2-1} = -\frac{9}{4 \sqrt{2}} \approx -1.786
\]

Therefore:

- the global minimum point is \( A(0.5, -1.786) \)
- the global maximum point is \( B(0, -1) \)

**Reading:** Nelson Textbook, Pages 130-134
**Homework:** Nelson Textbook: Page 136 #3ace, 4abe, 7ab, 10, 11