3.1 Higher Order Derivatives. Velocity and Acceleration

A Higher Order Derivatives

Let consider the function $y = f(x)$.

The *first* derivative of f or "f prime" is:

$$f'(x) = y' = \frac{dy}{dx}$$

The *second* derivative of f or "f double prime" is:

$$f''(x) = (f'(x))' = y'' = (y')' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

The *third* derivative of f or "f triple prime" is:

$$f'''(x) = (f''(x))' = y''' = (y'')' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The *fourth* derivative of f or "super 4" is:

$$f^{(4)}(x) = (f'''(x))' = y^{(4)} = (y''')' = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4}$$

The *n-th* derivative of f or "super n" is:

$$f^{(n)}(x) = (f^{(n-1)}(x))' = y^{(n)} = (y^{(n-1)})' = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$$

Notes.

1. f is *differentiable* at x if $f'(x)$ exists.
2. f is *double differentiable* at x if $f''(x)$ exists.
3. f is *n differentiable* at x if $f^{(n)}(x)$ exists.

Ex 1. Find f', f'', f''', $f^{(4)}$, and $f^{(5)}$ for $f(x) = -x^4 + 3x^2 - x + 1$.

$$f'(x) = -4x^3 + 6x - 1$$
$$f''(x) = -12x^2 + 6$$
$$f'''(x) = -24x$$
$$f^{(4)} = -24$$
$$f^{(5)} = 0$$
$$f^{(n)} = 0 \quad if \quad n \ge 5$$

Ex 2. Find f', f'', and f''' for $f(x) = \frac{x^3}{x^2+1}$.

$$f'(x) = \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - (x^4+3x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{4x^5+4x^3+6x^3+6x-4x^5-12x^3}{(x^2+1)^3} = \frac{-2x^3+6x}{(x^2+1)^3}$$

$$f'''(x) = \frac{(-6x^2+6)(x^2+1)^3 - (-2x^3+6x)(3)(x^2+1)^2(2x)}{(x^2+1)^6}$$

$$= \frac{-6x^4-6x^2+6x^2+6+12x^4-36x^2}{(x^2+1)^4}$$

$$= \frac{6x^4-36x^2+6}{(x^2+1)^4}$$

Ex 3. Find $f^{(n)}$ for $f(x) = \frac{x}{x+1}$.

$$f(x) = \frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1} = 1 - (x+1)^{-1}$$
$$f'(x) = -(-1)(x+1)^{-2}$$
$$f''(x) = -(-1)(-2)(x+1)^{-3}$$
$$f'''(x) = -(-1)(-2)(-3)(x+1)^{-4}$$
$$f^{(n)}(x) = -(-1)(-2)(-3)...(-n)(x+1)^{-(n+1)}$$
$$\therefore f^{(n)} = \frac{(-1)^{n+1}n!}{(x+1)^{n+1}}$$

Ex 4. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

$$y' = 3x^2 + 3$$
$$y'' = 6x$$
$$y''' = 6$$
$$LS = y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3)$$
$$= 6 + 6x^2 - 6x^2 - 6 = 0 = RS$$

B Velocity and Acceleration

Let $s(t)$ or $h(t)$ (height or altitude) be the *position function* of a particle.

$$\langle s \rangle_{IS} = \langle h \rangle_{IS} = m \text{ (meter)}$$

Displacement is the change in position over a time interval $[t_1, t_2]$:

$$\Delta s = s(t_2) - s(t_1) \qquad \Delta h = h(t_2) - h(t_1)$$

Average Velocity is defined by:

$$AV = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} \quad (m/s)$$

Velocity function is the derivative of the position function:

$$v(t) = s'(t) = h'(t)$$

Speed function is the absolute value of velocity $|v(t)|$.

Acceleration function is the derivative of velocity:

$$a(t) = v'(t) = s''(t) \quad (m/s^2)$$

Jerk function is the derivative of acceleration:

$$j(t) = a'(t) = v''(t) = s'''(t) \quad (m/s^3)$$

Notes:

$s(t) = 0$ (particle is in the origin)
$h(t) = 0$ (particle is on the ground)
$s(t) > 0$ (particle is on the positive side of the origin)
$s(t) < 0$ (particle is on the negative side of the origin)
$h(t) > 0$ (particle is above the ground)
$h(t) < 0$ (particle is below the ground)

$v(t) = 0$ (particle is at rest, changes direction of movement)
$v(t) > 0$ (particle moves in the positive direction (right or up))
$v(t) < 0$ (particle moves in the negative direction (left or down))

$a(t) = 0$ (particle is not accelerated)

If $|s|$ is increasing then the particle is *moving away from the origin* (MAFO) and $s(t)v(t) > 0$.
If $|s|$ is decreasing then the particle is *moving towards the origin* (MTO) and $s(t)v(t) < 0$.

If $|v|$ is increasing then the particle is *speeding up* (SU) and $v(t)a(t) > 0$.
If $|v|$ is decreasing then the particle is *slowing down* (SD) and $v(t)a(t) < 0$.

Ex 4. Consider the following position function

$$s(t) = \frac{t^3 - 12t}{4}.$$

a) Graph on the same grid $s(t)$, $v(t)$, $a(t)$.

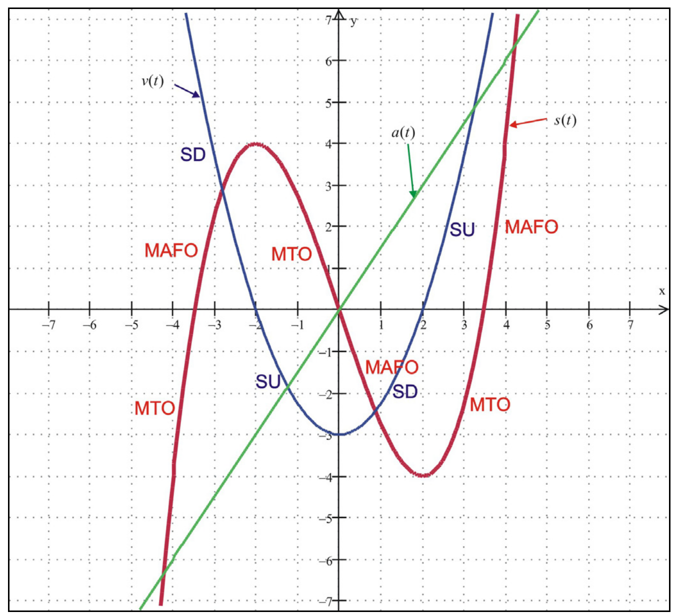
$s(t) = 0 \Rightarrow t(t^2 - 12) = 0 \Rightarrow t = 0 \ or \ t = \pm 2\sqrt{3}$

$v(t) = \frac{3t^2 - 12}{4}$

$v(t) = 0 \Rightarrow 3(t^2 - 4) = 0 \Rightarrow t = \pm 2 \Rightarrow s(\pm 2) = \mp 4$

$a(t) = \frac{6t}{4} = \frac{3}{2}t$

See the figure below:



b) Find the interval(s) when the particle is moving towards the origin.

According to the figure above, the particle is moving towards the origin (MTO) when $t \in (-\infty, -2\sqrt{3} \cup (-2, 0) \cup (2, 2\sqrt{3})$.

c) Find the interval(s) when the particle is speeding up.

According to the figure above the particle is speeding up (SU) when $t \in (-2, 0) \cup (2, \infty)$.

Homework:

$a)\ s(t) = (t^2 - 4)^2 \qquad b)\ s(t) = t^2(t^2 - 1)$
$c)\ s(t) = t^4 - 16 \qquad d)\ s(t) = t^3(t^2 - 1)$
$e)\ s(t) = t(t^4 - 1) \qquad f)\ s(t) = t^4 - 2t^2$

Reading: Nelson Textbook, Pages 119-126
Homework: Nelson Textbook: Page 127 #2def, 5, 3be, 10, 11, 14, 18