3.1 Higher Order Derivatives. Velocity and Acceleration

A Higher Order Derivatives
Let consider the function \( y = f(x) \).

The first derivative of \( f \) or “\( f \) prime” is:

\[ f'(x) = \frac{dy}{dx} \]

The second derivative of \( f \) or “\( f \) double prime” is:

\[ f''(x) = \frac{d}{dx}\left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \]

The third derivative of \( f \) or “\( f \) triple prime” is:

\[ f'''(x) = \frac{d}{dx}\left( \frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3} \]

The fourth derivative of \( f \) or “super 4” is:

\[ f^{(4)}(x) = \frac{d}{dx}\left( \frac{d^3y}{dx^3} \right) = \frac{d^4y}{dx^4} \]

The \( n \)-th derivative of \( f \) or “super \( n \)” is:

\[ f^{(n)}(x) = \frac{d}{dx}\left( \frac{d^{n-1}y}{dx^{n-1}} \right) = \frac{d^n y}{dx^n} \]

Notes.
1. \( f \) is differentiable at \( x \) if \( f'(x) \) exists.
2. \( f \) is double differentiable at \( x \) if \( f''(x) \) exists.
3. \( f \) is \( n \) differentiable at \( x \) if \( f^{(n)}(x) \) exists.

Ex 1. Find \( f' \), \( f'' \), \( f''' \), \( f^{(4)} \), and \( f^{(5)} \) for \( f(x) = -x^4 + 3x^2 - x + 1 \).

\[ f'(x) = -4x^3 + 6x - 1 \]
\[ f''(x) = -12x^2 + 6 \]
\[ f'''(x) = -24x \]
\[ f^{(4)}(x) = -24 \]
\[ f^{(5)}(x) = 0 \] if \( n \geq 5 \)

Ex 2. Find \( f' \), \( f'' \), and \( f''' \) for \( f(x) = \frac{x^3}{x^2 + 1} \).

\[ f'(x) = \frac{3x^2(x^2 + 1) - x^3(2x)}{(x^2 + 1)^2} = \frac{x^4 + 3x^2}{(x^2 + 1)^2} \]
\[ f''(x) = \frac{4x^3(3x) + 6x(x^2 + 1)^2 - (x^4 + 3x^2)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} \]
\[ = \frac{4x^5 + 4x^3 + 6x^3 + 6x - 4x^5 - 12x^3}{(x^2 + 1)^3} = - \frac{2x^3 + 6x}{(x^2 + 1)^3} \]
\[ f'''(x) = \frac{(-6x^2 + 6)(x^2 + 1)^3 - (-2x^3 + 6x)(3)(x^2 + 1)(2x)}{(x^2 + 1)^4} \]
\[ = \frac{-6x^4 - 6x^2 + 6x^2 + 12x^4 - 36x^2}{(x^2 + 1)^4} \]
\[ = \frac{6x^4 - 36x^2 + 6}{(x^2 + 1)^4} \]

Ex 3. Find \( f^{(n)} \) for \( f(x) = \frac{x}{x + 1} \).

\[ f(x) = \frac{x}{x + 1} = \frac{x + 1 - 1}{x + 1} = 1 - \frac{1}{x + 1} = 1 - (x + 1)^{-1} \]
\[ f'(x) = -(1)(x + 1)^{-2} \]
\[ f''(x) = -(-1)(-2)(x + 1)^{-3} \]
\[ f'''(x) = -(-1)(-2)(-3)(x + 1)^{-4} \]
\[ f^{(n)}(x) = -(-1)(-2)(-3)\ldots(-n)(x + 1)^{-n-1} \]
\[ \therefore f^{(n)}(x) = \frac{(-1)^n n!}{(x + 1)^{n+1}} \]

Ex 4. Show that \( y = x^3 + 3x + 1 \) satisfies \( y''' + xy'' - 2y' = 0 \).

\[ y' = 3x^2 + 3 \]
\[ y'' = 6x \]
\[ y''' = 6 \]
\[ LS = y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3) = 6 + 6x^2 - 6x^2 - 6 = 0 = RS \]
B Velocity and Acceleration

Let \( s(t) \) or \( h(t) \) (height or altitude) be the position function of a particle.

\[ s(t) = \text{position at time } t \]

Displacement is the change in position over a time interval \([t_1, t_2] \):

\[ \Delta s = s(t_2) - s(t_1) \quad \Delta h = h(t_2) - h(t_1) \]

Average Velocity is defined by:

\[ AV = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} \quad (m/s) \]

Velocity function is the derivative of the position function:

\[ v(t) = s'(t) = h'(t) \]

Speed function is the absolute value of velocity \( |v(t)| \).

Acceleration function is the derivative of velocity:

\[ a(t) = v'(t) = s''(t) \quad (m/s^2) \]

Jerk function is the derivative of acceleration:

\[ j(t) = a'(t) = v''(t) = s'''(t) \quad (m/s^3) \]

Notes:

\( s(t) = 0 \) (particle is in the origin)
\( h(t) = 0 \) (particle is on the ground)
\( s(t) > 0 \) (particle is on the positive side of the origin)
\( s(t) < 0 \) (particle is on the negative side of the origin)
\( h(t) > 0 \) (particle is above the ground)
\( h(t) < 0 \) (particle is below the ground)

\( v(t) = 0 \) (particle is at rest, changes direction of movement)
\( v(t) > 0 \) (particle moves in the positive direction (right or up))
\( v(t) < 0 \) (particle moves in the negative direction (left or down))

\( a(t) = 0 \) (particle is not accelerated)

If \( |s| \) is increasing then the particle is moving away from the origin (MAFO) and \( s(t)v(t) > 0 \).
If \( |s| \) is decreasing then the particle is moving towards the origin (MTO) and \( s(t)v(t) < 0 \).

If \( |v| \) is increasing then the particle is speeding up (SU) and \( v(t)a(t) > 0 \).
If \( |v| \) is decreasing then the particle is slowing down (SD) and \( v(t)a(t) < 0 \).

Ex 4. Consider the following position function

\[ s(t) = \frac{t^3 - 12t}{4} \]

a) Graph on the same grid \( s(t), v(t), a(t) \).

\[ s(t) = 0 \Rightarrow t(t^2 - 12) = 0 \Rightarrow t = 0 \text{ or } t = \pm 2\sqrt{3} \]

\[ v(t) = \frac{3t^2 - 12}{4} \]

\[ v(t) = 0 \Rightarrow 3(t^2 - 4) = 0 \Rightarrow t = \pm 2 \Rightarrow s(\pm 2) = \pm 4 \]

\[ a(t) = \frac{6t}{4} = \frac{3}{2} \]

See the figure below:

b) Find the interval(s) when the particle is moving towards the origin.

According to the figure above, the particle is moving towards the origin (MTO) when
\[ t \in (-\infty, -2\sqrt{3}) \cup (-2, 0) \cup (2, 2\sqrt{3}) \]

c) Find the interval(s) when the particle is speeding up.

According to the figure above the particle is speeding up (SU) when
\[ t \in (-2, 0) \cup (2, \infty) \]

Homework:

\( a) \) \( s(t) = (t^2 - 4)^2 \)  \( b) \) \( s(t) = t^2 (t^2 - 1) \)
\( c) \) \( s(t) = t^4 - 16 \)  \( d) \) \( s(t) = t^3 (t^2 - 1) \)
\( e) \) \( s(t) = t(t^4 - 1) \)  \( f) \) \( s(t) = t^4 - 2t^2 \)

Reading: Nelson Textbook, Pages 119-126

Homework: Nelson Textbook: Page 127 #2def, 5, 3be, 10, 11, 14, 18

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