2.4 Quotient Rule

A Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$ then so is $\frac{f}{g}$ and:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[f(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Proof:

$$\left(\frac{f}{g}\right)' = (fg^{-1})' = f'g^{-1} + f(-1)g^{-2}g'$$

$$= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}$$

Ex 1. Differentiate. Simplify the answer.

$$f(x) = \frac{x^2 - 1}{x^3 + 1}$$

$$f'(x) = \frac{(x^2-1)'(x^3+1) - (x^2-1)(x^3+1)'}{(x^3+1)^2}$$

$$= \frac{2x(x^3+1) - 3x^2(x^2-1)}{(x^3+1)^2} = \frac{2x^4 + 2x - 3x^4 + 3x^2}{(x^3+1)^2}$$

$$\therefore f'(x) = \frac{2x + 3x^2 - x^4}{(x^3+1)^2}$$

Ex 2. Given that $f(2) = 1$, $f'(2) = -1$, $g(2) = 2$, and $g'(2) = -2$ find $\left(\frac{g}{f+g}\right)'(2)$.

$$\left(\frac{g}{f+g}\right)' = \frac{g'(f+g) - g(f+g)'}{(f+g)^2} = \frac{g'(f+g) - g(f'+g')}{(f+g)^2}$$

$$\left(\frac{f}{f+g}\right)'(2) = \frac{g'(2)[f(2)+g(2)] - g(2)[f'(2)+g'(2)]}{[f(2)+g(2)]^2}$$

$$= \frac{-2(1+2) - 2(-1-2)}{(1+2)^2} = \frac{-6+6}{3^2} = 0$$

$$\therefore \left(\frac{f}{f+g}\right)'(2) = 0$$

Ex 3. Let $f(x) = \frac{x^3}{(1+x)^2}$. Find the point(s) on the graph of $y = f(x)$ where the tangent line is horizontal.

$$f'(x) = \frac{(x^3)'(1+x)^2 - (x^3)[(1+x)^2]'}{(1+x)^4}$$

$$= \frac{3x^2(1+x)^2 - (x^3)2(1+x)(1+x)'}{(1+x)^4}$$

$$= \frac{3x^2(1+x)^2 - 2(x^3)(1+x)}{(1+x)^4}$$

$$= \frac{x^2(1+x)[3(1+x) - 2x]}{(1+x)^4} = \frac{x^2(x+3)}{(1+x)^3}$$

$$f'(x) = 0 \quad \Rightarrow \quad \frac{x^2(x+3)}{(1+x)^3} = 0 \quad \Rightarrow \quad x^2(x+3) = 0$$

$$x = 0 \quad or \quad x = -3$$

$$x = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad \therefore A(0,0)$$

$$x = -3 \quad \Rightarrow \quad y = \frac{(-3)^3}{(1-3)^2} = -\frac{27}{4}$$

$$\therefore B(-3, -27/4)$$

Ex 4. Consider the position function $s(t)=\frac{\sqrt{t}}{t^2+1}, t\geq 0$.
Find the moment(s) of time when the particle is at rest.

The particle is at rest when $v(t)=s'(t)=0$.

$$v(t)=s'(t)=\frac{(\sqrt{t})'(t^2+1)-(\sqrt{t})(t^2+1)'}{(t^2+1)^2}$$

$$=\frac{\frac{1}{2\sqrt{t}}(t^2+1)-(\sqrt{t})(2t)}{(t^2+1)^2}$$

$$=\frac{\frac{1}{2\sqrt{t}}(t^2+1)-(\sqrt{t})(2t)\frac{2\sqrt{t}}{2\sqrt{t}}}{(t^2+1)^2}$$

$$=\frac{(t^2+1)-4t^2}{2\sqrt{t}(t^2+1)^2}=\frac{1-3t^2}{2\sqrt{t}(t^2+1)^2}$$

$$v(t)=0 \Rightarrow \frac{1-3t^2}{2\sqrt{t}(t^2+1)^2} \Rightarrow 1-3t^2=0 \Rightarrow t=\pm\frac{1}{\sqrt{3}}$$

Because $t\geq 0$, the particle is at rest when $t=\frac{1}{\sqrt{3}}$.

Ex 5. Let $y=f(x)=\frac{\sqrt[3]{x^2}}{x^2+1}$.

a) Differentiate. Simplify the answer.

$$f'(x)=\frac{(\sqrt[3]{x^2})'(x^2+1)-(\sqrt[3]{x^2})(x^2+1)'}{(x^2+1)^2}$$

$$=\frac{\frac{2}{3}x^{-\frac{1}{3}}(x^2+1)-x^{\frac{2}{3}}(2x)}{(x^2+1)^2}=\frac{\frac{2}{3}x^{-\frac{1}{3}}(x^2+1)-x^{\frac{2}{3}}x^{-\frac{1}{3}}x^{\frac{1}{3}}(2x)\frac{3}{3}}{(x^2+1)^2}$$

$$=\frac{2(x^2+1)-6x^2}{3x^{\frac{1}{3}}(x^2+1)^2}=\frac{2-4x^2}{3\sqrt[3]{x}(x^2+1)^2} \Rightarrow \therefore f'(x)=\frac{2-4x^2}{3\sqrt[3]{x}(x^2+1)^2}$$

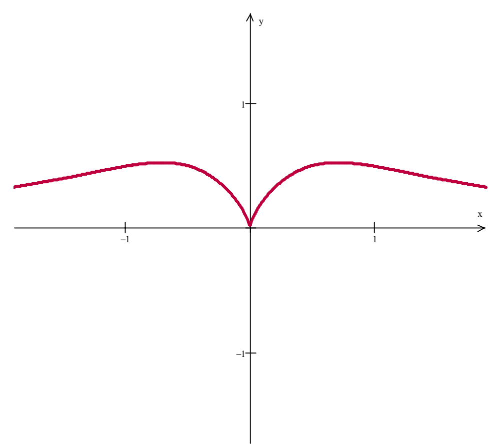
b) Find the points where the function is not differentiable.

The function is not differentiable at $O(0,0)$ (cusp point because $\lim_{x\to 0^-} f'(x)=-\infty$ and $\lim_{x\to 0^+} f'(x)=+\infty$).

c) Find the numbers x where the tangent line is horizontal.

$$f'(x)=\frac{2-4x^2}{3\sqrt[3]{x}(x^2+1)^2}=0 \Rightarrow 2-4x^2=0 \Rightarrow \therefore x=\pm\frac{1}{\sqrt{2}}$$

d) Use technology to graph the function.



Reading: Nelson Textbook, Pages 94-95
Homework: Nelson Textbook: Page 95 #4f, 5b, 8, 9a, 12, 14, 15, 16

