2.3 Product Rule

A Product Rule
If f and g are differentiable at x then so is fg and:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$(fg)' = f'g + fg'$$
$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$
$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$$

Ex 1. Use the product rule to prove the rule $(cf)' = cf'$ where c is a constant and f is a function.

$$(cf)' = c'f + cf' = 0(f) + cf' = cf'$$

Ex 2. Differentiate using the product rule: $y = (x^2+1)(2x^3 - x)$.

$$y' = (x^2+1)'(2x^3 - x) + (x^2+1)(2x^3 - x)'$$
$$= 2x(2x^3 - x) + (x^2+1)(6x^2 - 1)$$
$$\therefore y = 2x(2x^3 - x) + (x^2+1)(6x^2 - 1)$$

Ex 3. Find the equation of the tangent line to the curve $y = (x+\sqrt{x})\left(x^2 + \frac{1}{x}\right)$ at the point $P(1,4)$.

If $x = 1$ then $y = 4$. So $P(1,4)$ is on the graph.

$$\frac{dy}{dx} = (x+\sqrt{x})'\left(x^2+\frac{1}{x}\right) + (x+\sqrt{x})'\left(x^2+\frac{1}{x}\right)'$$
$$= \left(1+\frac{1}{2\sqrt{x}}\right)\left(x^2+\frac{1}{x}\right) + (x+\sqrt{x})\left(2x - \frac{1}{x^2}\right)$$
$$m = \left.\frac{dy}{dx}\right|_{x=1} = \left(1+\frac{1}{2\sqrt{1}}\right)\left(1^2+\frac{1}{1}\right) + (1+\sqrt{1})\left(2(1) - \frac{1}{1^2}\right)$$
$$= \frac{3}{2}(2) + 2(1) = 5$$
$$y - 4 = 5(x-1) \quad \Rightarrow \quad \therefore y = 5x - 1$$

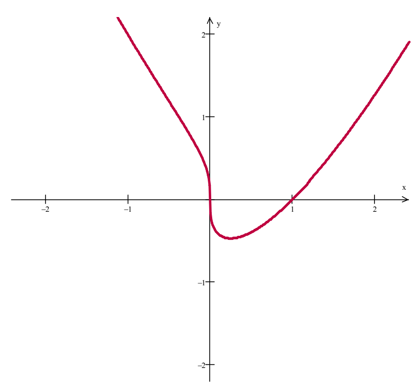
Ex 4. Consider the function $y = f(x) = \sqrt[3]{x}(x-1)$.

a) Differentiate and simplify.

$$y' = f'(x) = (\sqrt[3]{x})'(x-1) + (\sqrt[3]{x})(x-1)'$$
$$= \frac{1}{3}x^{-\frac{2}{3}}(x-1) + x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}}(x-1) + \frac{3}{3}x^{\frac{1}{3}}x^{-\frac{2}{3}}x^{\frac{2}{3}} = \frac{4x-1}{3\sqrt[3]{x^2}}$$
$$\therefore y' = \frac{4x-1}{3\sqrt[3]{x^2}}$$

b) Find the numbers where this function is not differentiable. Explain.

The function $y = f(x)$ is not differentiable at $x = 0$ because $y' = f'(x)$ does not exist at $x = 0$. There is a infinite slope point at $O(0,0)$ because $\lim_{x\to 0} f'(x) = -\infty$. See the figure bellow.



c) Find the numbers x where the tangent line is horizontal.

$$y' = 0 \quad \Rightarrow \quad \frac{4x-1}{3\sqrt[3]{x^2}} = 0 \quad \Rightarrow \quad 4x - 1 = 0$$
$$\therefore x = \frac{1}{4}$$

B Product of three functions If f, g and h are differentiable at x then so is fgh and: $\boxed{(fgh)' = f'gh + fg'h + fgh'}$ Proof: $(fgh)' = (fg)'h + fgh' = (f'g + fg')h + fgh'$ $= f'gh + fg'h + fgh'$	Ex 5. Differentiate. Do not simplify. $y = (\sqrt{x}+1)(x - x^2)\left(1 - \frac{1}{x}\right)$ $y' = (\sqrt{x}+1)'(x - x^2)\left(1 - \frac{1}{x}\right) + (\sqrt{x}+1)(x - x^2)'\left(1 - \frac{1}{x}\right) +$ $+ (\sqrt{x}+1)(x - x^2)\left(1 - \frac{1}{x}\right)'$ $y' = \frac{1}{2\sqrt{x}}(x - x^2)\left(1 - \frac{1}{x}\right) + (\sqrt{x}+1)(1-2x)\left(1 - \frac{1}{x}\right) + (\sqrt{x}+1)(x - x^2)\left(\frac{1}{x^2}\right)$
C Generalized Power Rule If f is differentiable at x, then so is f^n and: $([f(x)]^n)' = n[f(x)]^{n-1} f'(x)$ $(f^n)' = nf^{n-1} f'$ $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \frac{d}{dx} f(x)$ Proof for $n = 3$. $(fgh)' = f'gh + fg'h + fgh'$ If $f \equiv g \equiv h$ then: $(fff)' = f'ff + ff'f + fff'$ or $(f^3)' = 3f^2 f'$.	Ex 6. Differentiate $y = (x^2 - 3x)^{15}$. $y' = 15(x^2 - 3x)^{14}(x^2 - 3x)'$ $= 15(x^2 - 3x)^{14}(2x - 3)$ $\therefore y' = 15(x^2 - 3x)^{14}(2x - 3)$ Ex 7. Given $s(t) = t^2(2 - 3t)^3$ find the velocity at $t = 1$. $v(t) = s'(t) = (t^2)'(2 - 3t)^3 + (t^2)[(2 - 3t)^3]'$ $= 2t(2 - 3t)^3 + (t^2)(3)(2 - 3t)^2(2 - 3t)'$ $= 2t(2 - 3t)^3 + (t^2)(3)(2 - 3t)^2(-3)$ $v(1) = 2(1)(2 - 3(1))^3 + (1^2)(3)(2 - 3(1))^2(-3)$ $= 2(-1) + 3(1)(-3) = -11$ $\therefore v(1) = -11 m/s$

Reading: Nelson Textbook, Pages 85-90
Homework: Nelson Textbook: Page 91 #1b, 2d, 5e, 8b, 10, 12, 13

