2.2 Derivative of Polynomial Functions

A Power Rule
Consider the power function: \( y = x^n \), \( x, n \in \mathbb{R} \).
Then:

\[
(x^n)' = nx^{n-1}
\]

\[
\frac{d}{dx} x^n = nx^{n-1}
\]

Some useful specific cases:

\[
(1)' = 0
\]

\[
(x)' = 1
\]

\[
(\sqrt{x})' = \frac{1}{2\sqrt{x}}
\]

Ex 1. For each case, differentiate.

a) \((x^0)' = 0\)

b) \((x^1)' = 1\)

c) \((x^5)' = 5x^{5-1} = 5x^4\)

d) \(\left(\frac{1}{x}\right)' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}\)

e) \(\left(\frac{1}{x^7}\right)' = (-7)x^{-7-1} = -7x^{-8} = -\frac{7}{x^8}\)

f) \((x^2)' = 2x^{2-1} = 2x\)

\[
\frac{1}{2}x^2 = \frac{1}{2}x \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}\]

\[
\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}
\]

g) \((x^3)' = 3x^{3-1} = 3x^2\)

\[
\frac{1}{3}x^3 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}\]

\[
\frac{3}{3\sqrt{x}} = \frac{3}{3\sqrt{x}}
\]

h) \((x^4)' = 4x^{4-1} = 4x^3\)

\[
\frac{3}{4}x^4 = \frac{3}{4} \cdot \frac{3}{4} = \frac{3}{4}\]

\[
\frac{3}{4\sqrt{x}} = \frac{3}{4\sqrt{x}}
\]

i) \((x^\pi)' = \pi x^{\pi-1}\)

B Constant Function Rule
Let consider a constant function: \( f(x) = c, \ c \in \mathbb{R} \).
Then:

\[
(c)' = 0
\]

\[
\frac{d}{dx} c = 0
\]

Ex 2. Find each derivative function:

a) \((-2)' = 0\)

b) \((\sqrt{x})' = 0\)

C Constant Multiple Rule
Let consider \( g(x) = cf(x) \). Then:

\[
[cf(x)]' = cf'(x)
\]

\[
\frac{d}{dx}[cf(x)] = c \cdot \frac{d}{dx} f(x)
\]

\[
(cf)' = cf'
\]

Ex 3. Differentiate each expression:

a) \(-2x^3\)

\[
(-2x^3)' = (-2)(x^3)' = (-2)(3)x^{3-1} = -6x^2
\]

b) \(-3x^2\)

\[
\left(-\frac{3}{x^2}\right)' = (-3x^{-2})' = (-3)(-2)x^{-2-1} = 6x^{-3} = 6
\]

\[
\frac{1}{x^3} = \frac{1}{x^3}
\]

c) \(\sqrt{2x}\)

\[
(\sqrt{2x})' = \sqrt{2}(\sqrt{x})' = \sqrt{2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2x}}
\]

D Sum and Difference Rules

\[
[f(x) \pm g(x)] = f'(x) \pm g'(x)
\]

\[
\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)
\]

\[
(f \pm g)' = f' \pm g'
\]

Ex 4. Differentiate.

a) \(f(x) = 2 - 3x + 4x^2 - 3x^7\)

\[
f'(x) = (2 - 3x + 4x^2 - 3x^7)' = (2)' + (-3x)' + (4x^2)' + (-3x^7)' = 0 + (-3)x + 4(2x^1) - 3(7x^6)
\]

\[
= -3 + 4(2x) - 3(7)x^6 = -3 + 8x - 21x^6
\]

\[
\therefore f'(x) = -3 + 8x - 21x^6
\]
b) \( g(x) = \frac{1}{x} - \frac{2}{x^2} + \frac{4}{x^5} \)

\[
g'(x) = \left( \frac{1}{x} - \frac{2}{x^2} + \frac{4}{x^5} \right)' = (x^{-1} - 2x^{-2} + 4x^{-5})' = (x^{-1})' + (-2x^{-2})' + (4x^{-5})' = -1(x^{-2}) + (-2)(-2)x^{-3} + 4(-5)x^{-6} = -\frac{1}{x^2} + \frac{4}{x^3} - \frac{20}{x^6}.
\]

\[
\therefore g'(x) = -\frac{1}{x^2} + \frac{4}{x^3} - \frac{20}{x^6}.
\]

c) \( h(x) = \frac{x - 3\sqrt{x}}{x} \)

\[
h'(x) = \left( \frac{x - 3\sqrt{x}}{x} \right)' = \left( \frac{x}{x} \right)' - 3\left( \frac{1}{2\sqrt{x}} \right)x = 1 - \frac{3}{2\sqrt{x}}.
\]

\[
\therefore h'(x) = 1 - \frac{3}{2\sqrt{x}}.
\]

---

**E Tangent Line**

To find the equation of the tangent line at the point \( P(a, f(a)) \):

1. Find derivative function \( f'(a) \).
2. Find the slope of the tangent line using: \( m = f'(a) \).
3. Use the slope-point formula to get the equation of the tangent line: \( y - f(a) = m(x-a) \).

---

**Ex 5.** Find the equation of the tangent line at the point \( P(1,0) \) to the graph of \( y = f(x) = x^2 - \frac{1}{x} \).

\[
f'(x) = 2x + \frac{1}{x^2}
\]

\[
m = f'(1) = 2(1) + 1/1^2 = 3
\]

\[
y - 0 = 3(x-1)
\]

\[
\therefore y = 3x - 3
\]

---

**Ex 6.** Find the equation of the tangent line of the slope \( m = 2 \) to the graph of \( y = f(x) = 2\sqrt{x} \). Graph the function and the tangent line.

\[
f'(x) = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}
\]

\[
f'(x) = 2 \quad \Rightarrow \quad \frac{1}{\sqrt{x}} = 2 \quad \Rightarrow \quad x = \frac{1}{4} \quad \Rightarrow \quad y = 2\sqrt{\frac{1}{4}} = 1
\]

\[
P(1/4,1) \quad \Rightarrow \quad y-1 = 2(x-1/4)
\]

\[
\therefore y = 2x + 1/2
\]

---

**Ex 7.** Find the points on the graph of \( y = f(x) = 2x^3 - 3x^2 + 1 \) where the tangent line is horizontal.

\[
f'(x) = 6x^2 - 6x = 6x(x-1)
\]

\[
f'(x) = m
\]

\[
m = 0 \text{ (horizontal tangent)}
\]

\[
6x(x-1) = 0
\]

\[
x = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad A(0,1)
\]

\[
x = 1 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad B(1,0)
\]

The tangent line is horizontal at \( A(0,1) \) and \( B(1,0) \).
F Normal Line

If \( m_T \) is the slope of the tangent line, then slope of the normal line \( m_N \) is given by:

\[
m_N = -\frac{1}{m_T}
\]

Ex 8. Find the equation of the normal line to the curve \( y = f(x) = x + \frac{2}{x} \) at \( P(1,3) \).

\[
f'(x) = 1 - \frac{2}{x^2}
\]

\[
m_T = f'(1) = -1
\]

\[
m_N = -\frac{1}{m_T} = 1
\]

\[
y - 3 = 1(x - 1)
\]

\[
\therefore y = x + 2
\]

Ex 9. Analyze the differentiability of each function.

a) \( y = f(x) = \sqrt[3]{x} \)

The function \( f \) is continuous over \( R \).

\[
f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3}x^{-\frac{1}{3}} = \frac{1}{3\sqrt[3]{x^2}}
\]

\( f'(x) \) does not exist at \( x = 0 \). Therefore the function \( f \) is not differentiable at \( x = 0 \).

As \( x \to 0^- \), \( f'(x) \to -\infty \). The point \( O(0,0) \) is an infinite slope point.

b) \( y = f(x) = x^{\frac{2}{3}} \)

The function \( f \) is continuous over \( R \).

\[
f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}
\]

\( f'(x) \) does not exist at \( x = 0 \).

The function \( f \) is not differentiable at \( x = 0 \).

As \( x \to 0^- \), \( f'(x) \to -\infty \).

As \( x \to 0^+ \), \( f'(x) \to \infty \).

The point \( O(0,0) \) is a cusp point.

c) \( y = f(x) = |x - 3| \)

\[
f(x) = \begin{cases} x - 3, & x \geq 3 \\ 3 - x, & x < 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 1, & x > 3 \\ -1, & x < 3 \end{cases}
\]

\[
\lim_{x \to 3^-} f'(x) = -1 \neq \lim_{x \to 3^+} f'(x) = 1
\]

\( f' \) does not exist at \( x = 3 \). Therefore, \( f \) is not differentiable at \( x = 3 \). The point \( P(3,0) \) is a corner point (see the figure to the right side).
H Differentiability for piece-wise defined functions
Let consider the piece-wise defined function:
\[
    f(x) = \begin{cases} 
    f_1(x), & x < a \\
    c, & x = a \\
    f_2(x), & x > a 
    \end{cases}
\]

The function \( f \) is differentiable at \( x = a \) if:
(a) the function is continuous at \( x = a \)
(b) \( f'(a) = f_2'(a) \) (the slope of the tangent line for the left branch is equal to the slope of the tangent line for the right branch).

Ex 10. Analyze the differentiability of each function at \( x = 1 \).

a) \( f(x) = \begin{cases} 
    x^2, & x \leq 1 \\
    2x, & x > 1 
    \end{cases} \)

\[
    \lim_{{x \to 1^-}} f(x) = 1, \quad \lim_{{x \to 1^+}} f(x) = 2, \quad f(1) = 1
\]

The function \( f \) is not continuous at \( x = 1 \). Therefore, the function \( f \) is not differentiable at \( x = 1 \). See the figure below.

---

b) a) \( f(x) = \begin{cases} 
    x^2, & x \leq 1 \\
    x, & x > 1 
    \end{cases} \)

\[
    \lim_{{x \to 1^-}} f(x) = 1, \quad \lim_{{x \to 1^+}} f(x) = 1, \quad f(1) = 1
\]

The function is continuous at \( x = 1 \).

\[
    f'(x) = \begin{cases} 
    2x, & x < 1 \\
    1, & x > 1 
    \end{cases}
\]

\[
    \lim_{{x \to 1^-}} f'(x) = 2, \quad \lim_{{x \to 1^+}} f(x) = 1
\]

The function \( f \) is not differentiable at \( x = 1 \) (corner point). See the figure below.

c) a) \( f(x) = \begin{cases} 
    x^2, & x \leq 1 \\
    2x - 1, & x > 1 
    \end{cases} \)

\[
    \lim_{{x \to 1^-}} f(x) = 1, \quad \lim_{{x \to 1^+}} f(x) = 1, \quad f(1) = 1
\]

The function is continuous at \( x = 1 \).

\[
    f'(x) = \begin{cases} 
    2x, & x < 1 \\
    2, & x > 1 
    \end{cases}
\]

\[
    \lim_{{x \to 1^-}} f'(x) = 2, \quad \lim_{{x \to 1^+}} f(x) = 2
\]

The function \( f \) is differentiable at \( x = 1 \). See the figure below.

---

Reading: Nelson Textbook, Pages 76-81
Homework: Nelson Textbook: Page 82 #2cdf, 3ce, 4aef, 5b, 6b, 7a, 8b, 9b, 11, 14, 17, 20, 25iii, 28