2.2 Derivative of Polynomial Functions

A Power Rule Consider the *power* function: $y = x^n, \quad x, n \in R$. Then: $(x^n)' = nx^{n-1}$ $\frac{d}{dx} x^n = nx^{n-1}$ Some useful specific cases: $(1)' = 0$ $(x)' = 1$ $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	Ex 1. For each case, differentiate. a) $(1)' = (x^0)' = 0x^{0-1} = 0$ b) $(x)' = (x^1)' = 1(x^{1-1}) = 1(x^0) = 1$ c) $(x^5)' = 5(x^{5-1}) = 5x^4$ d) $\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$ e) $\left(\frac{1}{x^7}\right)' = (x^{-7})' = (-7)x^{-7-1} = -7x^{-8} = \frac{-7}{x^8}$ f) $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$ g) $(\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$ h) $(\sqrt[4]{x^3})' = (x^{\frac{3}{4}})' = \frac{3}{4}x^{\frac{3}{4}-1} = \frac{3}{4}x^{\frac{-1}{4}} = \frac{3}{4x^{\frac{1}{4}}} = \frac{3}{4\sqrt[4]{x}}$ i) $(x^{\pi})' = \pi x^{\pi-1}$
B Constant Function Rule Let consider a *constant* function: $f(x) = c, \quad c \in R$. Then: $(c)' = 0$ $\frac{d}{dx} c = 0$	Ex 2. Find each derivative function: a) $(-2)' = 0$ b) $(\sqrt{\pi})' = 0$
C Constant Multiple Rule Let consider $g(x) = cf(x)$. Then: $[cf(x)]' = cf'(x)$ $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$ $(cf)' = cf'$	Ex 3. Differentiate each expression: a) $-2x^3$ $(-2x^3)' = (-2)(x^3)' = (-2)(3)x^{3-1} = -6x^2$ b) $\frac{-3}{x^2}$ $\left(\frac{-3}{x^2}\right)' = (-3x^{-2})' = (-3)(x^{-2})' = (-3)(-2)x^{-2-1} = 6x^{-3} = \frac{6}{x^3}$ c) $\sqrt{2x}$ $(\sqrt{2x})' = \sqrt{2}(\sqrt{x})' = \sqrt{2}\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2x}}$
D Sum and Difference Rules $[f(x) \pm g(x)] = f'(x) \pm g'(x)$ $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ $(f \pm g)' = f' \pm g'$	Ex 4. Differentiate. a) $f(x) = 2 - 3x + 4x^2 - 3x^7$ $f'(x) = (2 - 3x + 4x^2 - 3x^7)'$ $= (2)' + (-3x)' + (4x^2)' + (-3x^7)'$ $= 0 + (-3)(x)' + 4(x^2)' - 3(x^7)'$ $= -3 + 4(2)x - 3(7)x^6$ $= -3 + 8x - 21x^6$ $\therefore f'(x) = -3 + 8x - 21x^6$

b) $g(x)=\frac{1}{x}-\frac{2}{x^2}+\frac{4}{x^5}$

$g'(x)=\left(\frac{1}{x}-\frac{2}{x^2}+\frac{4}{x^5}\right)'=(x^{-1}-2x^{-2}+4x^{-5})'$

$=(x^{-1})'+(-2x^{-2})'+(4x^{-5})'$

$=-1(x^{-2})+(-2)(-2)x^{-3}+4(-5)x^{-6}$

$=-\frac{1}{x^2}+\frac{4}{x^3}-\frac{20}{x^6}$

$\therefore g'(x)=-\frac{1}{x^2}+\frac{4}{x^3}-\frac{20}{x^6}$

c) $h(x)=\frac{\sqrt{x}-\sqrt[3]{x}}{x}$

$h'(x)=\left(\frac{\sqrt{x}-\sqrt[3]{x}}{x}\right)'=\left(\frac{\sqrt{x}}{x}-\frac{\sqrt[3]{x}}{x}\right)'$

$=(x^{\frac{1}{2}-1}-x^{\frac{1}{3}-1})'=(x^{-\frac{1}{2}}-x^{\frac{-2}{3}})'=(x^{-\frac{1}{2}})'-(x^{\frac{-2}{3}})'$

$=\left(\frac{-1}{2}\right)x^{\frac{-1}{2}-1}-\left(\frac{-2}{3}\right)x^{\frac{-2}{3}-1}=\left(\frac{-1}{2}\right)x^{\frac{-3}{2}}-\left(\frac{-2}{3}\right)x^{\frac{-5}{3}}$

$=\frac{-1}{2\sqrt{x^3}}+\frac{2}{3\sqrt[3]{x^5}}=-\frac{1}{2x\sqrt{x}}+\frac{2}{3x\sqrt[3]{x^2}}$

$\therefore h'(x)=-\frac{1}{2x\sqrt{x}}+\frac{2}{3x\sqrt[3]{x^2}}$

E Tangent Line

To find the *equation of the tangent line* at the point $P(a.f(a))$:

1. Find *derivative* function $f'(x)$.
2. Find the *slope* of the tangent line using:

$$m=f'(a)$$

3. Use the *slope-point* formula to get the equation of the tangent line:

$$y-f(a)=m(x-a)$$

Ex 5. Find the equation of the tangent line at the point $P(1,0)$ to the graph of $y=f(x)=x^2-\frac{1}{x}$.

$f'(x)=2x+\frac{1}{x^2}$

$m=f(1)=2(1)+1/1^2=3$

$y-0=3(x-1)$

$\therefore y=3x-3$

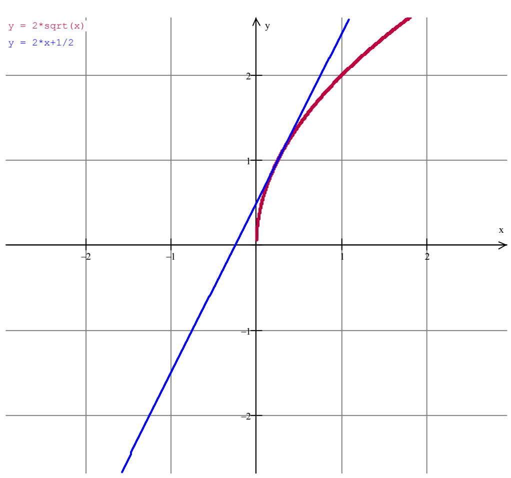
Ex 6. Find the equation of the tangent line of the slope $m=2$ to the graph of $y=f(x)=2\sqrt{x}$. Graph the function and the tangent line.

$f'(x)=2\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}}$

$f'(x)=2 \Rightarrow \frac{1}{\sqrt{x}}=2 \Rightarrow x=\frac{1}{4} \Rightarrow y=2\sqrt{\frac{1}{4}}=1$

$P(1/4,1) \Rightarrow y-1=2(x-1/4)$

$\therefore y=2x+1/2$



Ex 7. Find the points on the graph of $y=f(x)=2x^3-3x^2+1$ where the tangent line is horizontal.

$f'(x)=6x^2-6x=6x(x-1)$

$f'(x)=m$

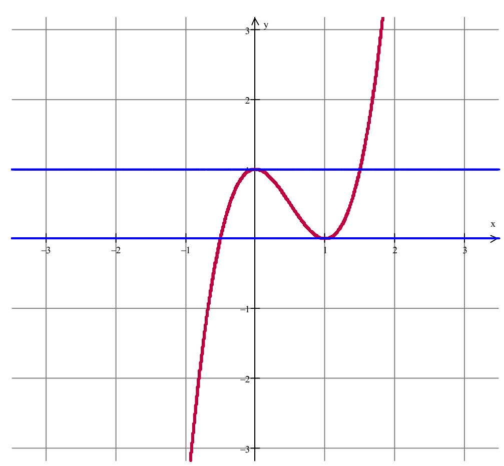
$m=0$ (horizontal tangent)

$6x(x-1)=0$

$x=0 \Rightarrow y=1 \Rightarrow A(0,1)$

$x=1 \Rightarrow y=0 \Rightarrow B(1,0)$

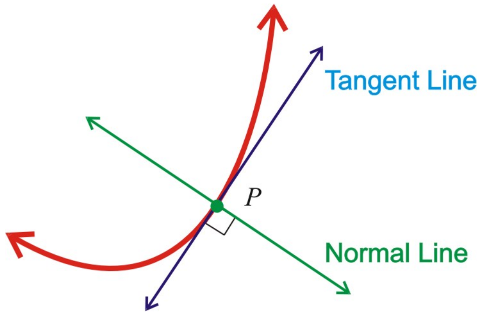
The tangent line is horizontal at $A(0,1)$ and $B(1,0)$.



F Normal Line

If m_T is the slope of the tangent line, then slope of the normal line m_N is given by:

$$m_N = -\frac{1}{m_T}$$



Ex 8. Find the equation of the normal line to the curve $y = f(x) = x + \frac{2}{x}$ at $P(1,3)$.

$f'(x) = 1 - \frac{2}{x^2}$

$m_T = f(1) = -1$

$m_N = -\frac{1}{m_T} = 1$

$y - 3 = 1(x-1)$

$\therefore y = x + 2$

Ex 9. Analyze the differentiability of each function.

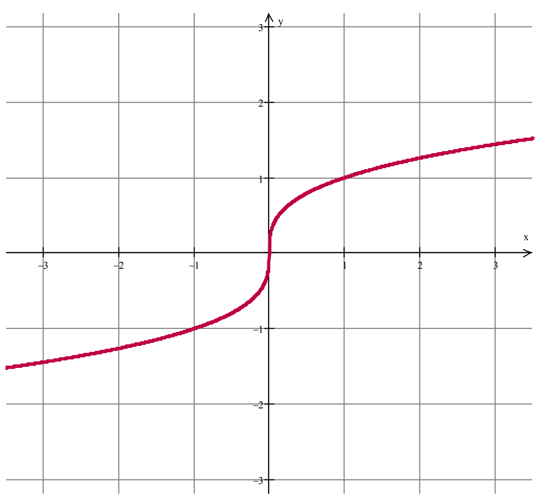
a) $y = f(x) = \sqrt[3]{x}$

The function f is continuous over R.

$f'(x) = (x^{\frac{1}{3}})' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$f'(x)$ does not exist at $x = 0$. Therefore the function f is not differentiable at $x = 0$.

As $x \to 0$, $f'(x) \to \infty$. The point $O(0,0)$ is a infinite slope point.



b) $y = f(x) = x^{2/3}$

The function f is continuous over R.

$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

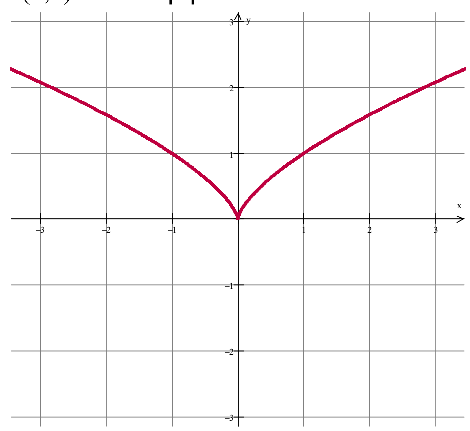
$f'(x)$ does not exist at $x = 0$.

The function f is not differentiable at $x = 0$.

As $x \to 0^-$, $f'(x) \to -\infty$.

As $x \to 0^+$, $f'(x) \to \infty$.

The point $O(0,0)$ is a cusp point.

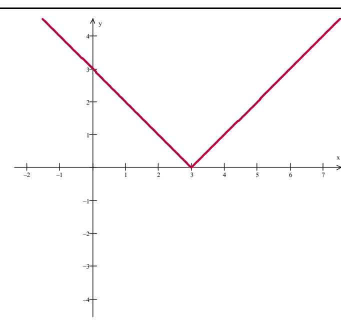


c) $y = f(x) = |x-3|$

$$f(x) = \begin{cases} x-3, & x \ge 3 \\ 3-x, & x < 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 1, & x > 3 \\ -1, & x < 3 \end{cases}$$

$\lim_{x \to 3^-} f'(x) = -1 \quad \ne \quad \lim_{x \to 3^+} f'(x) = 1$

f' does not exist at $x = 3$. Therefore, f is not differentiable at $x = 3$. The point $P(3,0)$ is a corner point (see the figure to the right side).



H Differentiability for piece-wise defined functions

Let consider the piece-wise defined function:

$$f(x)=\begin{cases} f_1(x), & x<a \\ c, & x=a \\ f_2(x), & x>a \end{cases}$$

The function f is *differentiable* at $x=a$ if:

(a) the function is continuous at $x=a$

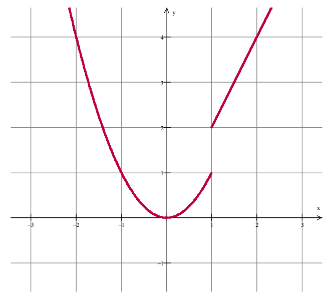
(b) $f_1'(a)=f_2'(a)$ (the slope of the tangent line for the left branch is equal to the slope of the tangent line for the right branch).

Ex 10. Analyze the differentiability of each function at $x=1$.

a) $f(x)=\begin{cases} x^2, & x\le 1 \\ 2x, & x>1 \end{cases}$

$\lim_{x\to 1^-} f(x)=1,\quad \lim_{x\to 1^+} f(x)=2,\quad f(1)=1$

The function f is not continuous at $x=1$. Therefore, the function f is not differentiable at $x=1$. See the figure below.



b) a) $f(x)=\begin{cases} x^2, & x\le 1 \\ x, & x>1 \end{cases}$

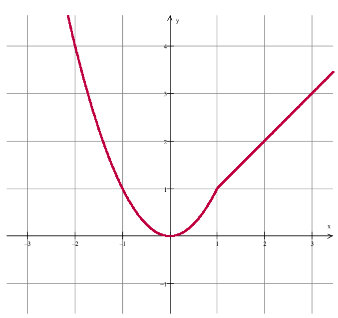
$\lim_{x\to 1^-} f(x)=1,\quad \lim_{x\to 1^+} f(x)=1,\quad f(1)=1$

The function is continuous at $x=1$.

$f'(x)=\begin{cases} 2x, & x<1 \\ 1, & x>1 \end{cases}$

$\lim_{x\to 1^-} f'(x)=2,\quad \lim_{x\to 1^+} f(x)=1$

The function f is not differentiable at $x=1$ (corner point). See the figure below.



c) a) $f(x)=\begin{cases} x^2, & x\le 1 \\ 2x-1, & x>1 \end{cases}$

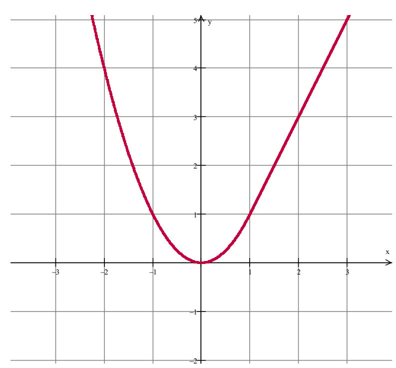
$\lim_{x\to 1^-} f(x)=1,\quad \lim_{x\to 1^+} f(x)=1,\quad f(1)=1$

The function is continuous at $x=1$.

$f'(x)=\begin{cases} 2x, & x<1 \\ 2, & x>1 \end{cases}$

$\lim_{x\to 1^-} f'(x)=2,\quad \lim_{x\to 1^+} f(x)=2$

The function f is differentiable at $x=1$. See the figure below.



Reading: Nelson Textbook, Pages 76-81
Homework: Nelson Textbook: Page 82 #2cdf, 3ce, 4aef, 5b, 6b, 7a, 8b, 9b, 11, 14, 17, 20, 25iii, 28

