1.6 Continuity

A Continuity

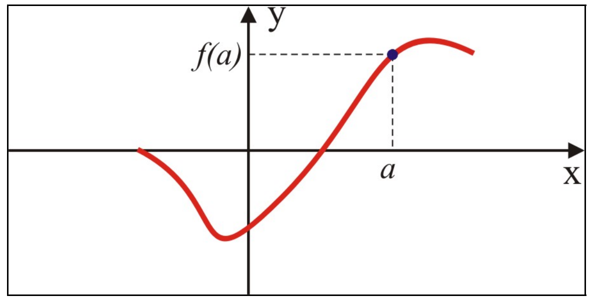
A function $y = f(x)$ is *continuous* at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Note: A function is continuous at a if the following three conditions are met:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $f(a)$ and $\lim_{x \to a} f(x)$ are equal.

Note: A function is continuous if the graph can be drawn *without lifting* the pen from paper.



B Discontinuity

If $y = f(x)$ is not continuous at a then we say:

- $y = f(x)$ is *discontinuous* at a or
- $y = f(x)$ has a discontinuity at a

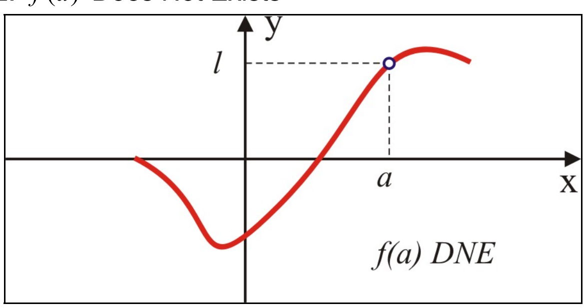
Note: There are three types of discontinuity:

a) *removable* or *point* discontinuity
b) *jump* discontinuity
c) *infinite* discontinuity (break)

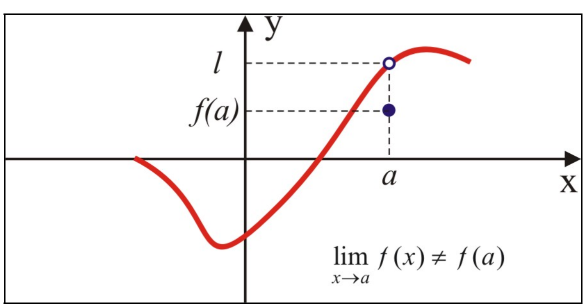
C Removable or Point Discontinuity

A function $y = f(x)$ has a *removable* or *point discontinuity* at a if:

1. $\lim_{x \to a} f(x)$ exists
2. $f(a)$ Does Not Exists



or $\lim_{x \to a} f(x) \neq f(a)$



Note: A removable or point discontinuity *can be removed* by redefining the function at a as

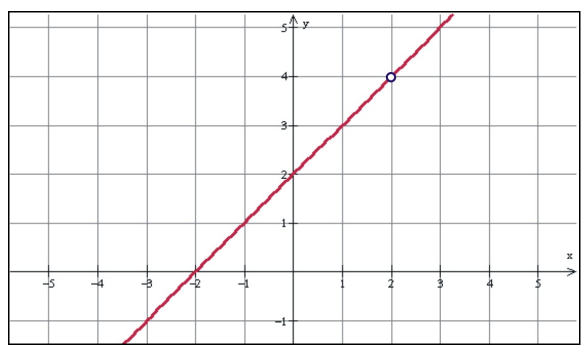
$$f(a) \overset{def}{=} \lim_{x \to a} f(x).$$

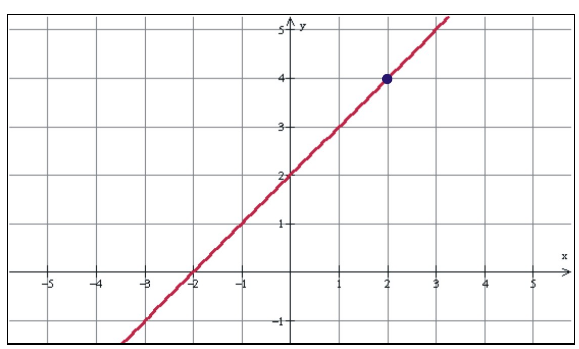
Ex 1. Redefine $y = f(x) = \dfrac{x^2 - 4}{x - 2}$ such that $y = f(x)$ is to be continuous everywhere (at any number). Graph the old and the new function.

$f(2)$ does not exist.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2}(x+2) = 4$$

$$\therefore g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} = x + 2, & x \neq 2 \\ 4, & x = 2 \end{cases}$$ is continuous everywhere.

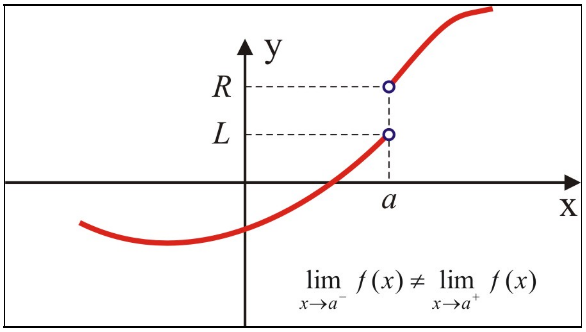




D Jump Discontinuity

A function $y = f(x)$ has a *jump discontinuity* at a if the left-side and the right-side limits exist but they are not equal:

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

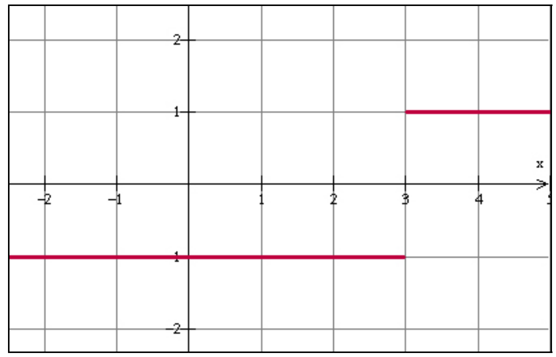


Ex 2. Analyze the continuity of the function $y = f(x) = \frac{|x-3|}{x-3}$ at $x = 3$. Graph the function to illustrate the situation.

$$y = f(x) = \frac{|x-3|}{x-3} = \begin{cases} \frac{x-3}{x-3} = 1, & x > 3 \\ \frac{3-x}{x-3} = -1, & x < 3 \end{cases}$$

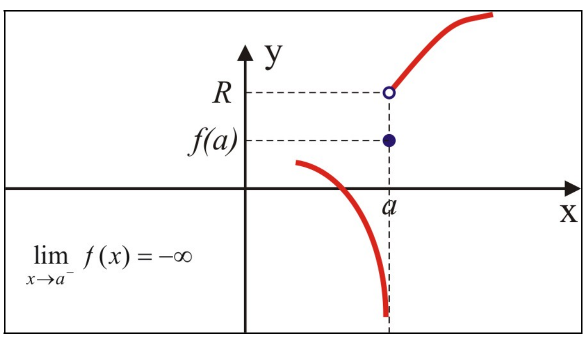
$$\lim_{x \to 3^-} f(x) = -1 \neq \lim_{x \to 3^+} f(x) = 1$$

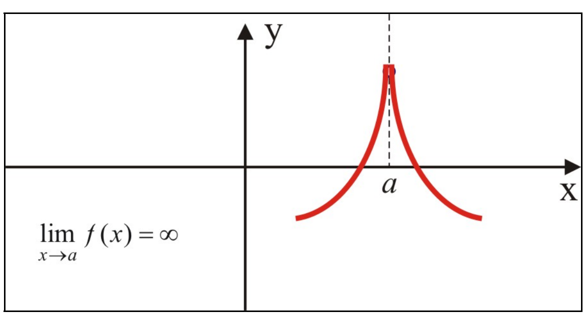
Therefore, the function has a jump discontinuity at $x = 3$.



E Infinite Discontinuity

A function $y = f(x)$ has an *infinite discontinuity* at a if at least one of the left-side or the right-side limits is *unbounded* (approaches to ∞ or $-\infty$).

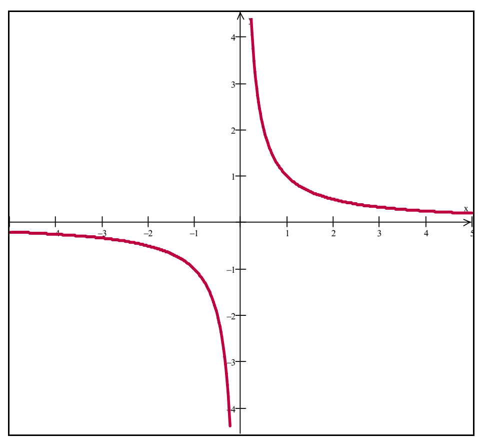




To write $\lim_{x \to a} f(x) = \infty$ is better (more information is included) than to write $\lim_{x \to a} f(x)$ DNE.

Ex 3. Analyze the continuity of the function $f(x) = \frac{1}{x}$ at $x = 0$.

The graph of the function $f(x) = \frac{1}{x}$ is represented in the figure below:



$$\lim_{x \to 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \to 0^+} \frac{1}{x} = \infty \quad \Rightarrow \quad \lim_{x \to 0} \frac{1}{x} = DNE$$

The function $f(x) = \frac{1}{x}$ has an infinite discontinuity at $x = 0$.

Ex 4. The function $y = f(x)$ is represented graphically in the figure on the right side.

Analyze the continuity of this function at:

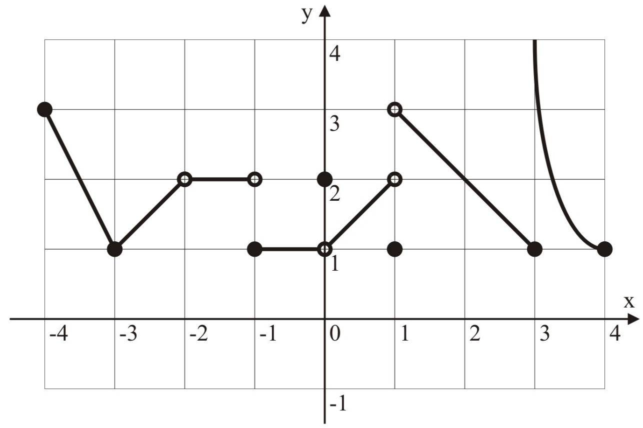
a) $x = -3$ (continuous)

b) $x = -2$ (discontinuous, removable discontinuity)

c) $x = -1$ (discontinuous, jump discontinuity)

d) $x = 0$ (discontinuous, removable discontinuity)

e) $x = 3$ (discontinuous, infinite discontinuity)



F Elementary Functions
Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are *continuous* over their domain.

Ex 5. Analyze the continuity of the function:

$$f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ x^3 + 1, & x > 1 \end{cases}$$

The function $f(x)$ is continuous over $(-\infty, 0)$, $(0,1)$ and $(1, \infty)$ (because $y = x$, $y = x^2$, and $y = x^3 + 1$ are elementary functions).

Let analyze the continuity at $x = 0$:

$$\left.\begin{array}{l} \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0 \\ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0 \\ f(0) = 0^2 = 0 \end{array}\right\} \Rightarrow f(x) \text{ is continuous at}$$

$x = 0$.

Let analyze the continuity at $x = 1$:

$$\left.\begin{array}{l} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1 \\ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^3 + 1) = 1^3 + 1 = 2 \\ f(1) = 1^2 = 1 \end{array}\right\} \Rightarrow f(x) \text{ is}$$

discontinuous at $x = 1$ (jump discontinuity).

Therefore, the function $f(x)$ is discontinuous at $x = 1$ and continuous elsewhere.

Ex 6. For what value of the constant c is the function

$$f(x) = \begin{cases} x + c, & x < 2 \\ cx^2 + 1, & x \ge 2 \end{cases}$$

continuous at any number (everywhere)?

The function $f(x)$ is continuous over $(-\infty, 2)$ and $(2, \infty)$ (because $y = x + c$ and $y = cx^2 + 1$ are elementary functions).

Let analyze the continuity at $x = 2$:

$$\left.\begin{array}{l} \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + c) = 2 + c \\ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (cx^2 + 1) = c(2^2) + 1 = 4c + 1 \\ f(2) = c(2^2) + 1 = 4c + 1 \end{array}\right\} \Rightarrow f(x) \text{ is}$$

continuous at $x = 2$ if $2 + c = 4c + 1$ or $c = \frac{1}{3}$.

Therefore, $f(x)$ is continuous everywhere if $c = \frac{1}{3}$.

Reading: Nelson Textbook, Pages 48-51
Homework: Nelson Textbook: Page 51 #4a, 5c, 7, 12, 15, 16, 17