1.5 Properties of Limits

A Limits Properties We assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then: 1. $\lim_{x\to a} k = k$ 2. $\lim_{x\to a} x = a$ 3. $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ 4. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ 5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \lim_{x\to a} g(x) \neq 0$ 6. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$ 7. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x]^n$ 8. If $P(x)$ is a polynomial function, then $\lim_{x\to a} P(x) = P(a)$	Ex 1. Given $\lim_{x\to 3} f(x) = -2$ and $\lim_{x\to 3} g(x) = 1$, use the limits properties to find $\lim_{x\to 3}\frac{2f(x)+g(x)}{-4\sqrt{g(x)}}$. $\lim_{x\to 3}\frac{2f(x)+g(x)}{-4\sqrt{g(x)}} = \frac{\lim_{x\to 3}[2f(x)+g(x)]}{\lim_{x\to 3}[-4\sqrt{g(x)}]} =$ $\frac{\lim_{x\to 3}[2f(x)]+\lim_{x\to 3} g(x)}{-4\lim_{x\to 3}\sqrt{g(x)}} = \frac{2\lim_{x\to 3} f(x)+\lim_{x\to 3} g(x)}{-4\sqrt{\lim_{x\to 3} g(x)}} =$ $= \frac{2(-2)+1}{-4\sqrt{1}} = \frac{-4+1}{-4} = \frac{3}{4}$ $\therefore \lim_{x\to 3}\frac{2f(x)+g(x)}{-4\sqrt{g(x)}} = \frac{3}{4}$
B Substitution Substitution is the best strategy to find a limit. Note: Substitution does not work if (by substitution) you get one of the following 7 *indeterminate cases*: $\infty-\infty \quad 0\times\infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^\infty \quad \infty^0 \quad 0^0$ Specific strategies are available to avoid an indeterminate case to appear.	Ex 2. Compute $\lim_{x\to 2}\frac{x^2-2x+1}{x-1}$. $\lim_{x\to 2}\frac{x^2-2x+1}{x-1} = \frac{2^2-2(2)+1}{2-1} = \frac{4-4+1}{1} = 1$ $\therefore \lim_{x\to 2}\frac{x^2-2x+1}{x-1} = 1$
C Factoring The indeterminate form $\frac{0}{0}$ may be eliminated by factoring and *canceling out the common factor* that generates zeros: $\lim_{x\to a}\frac{F(x)}{G(x)} = \lim_{x\to a}\frac{(x-a)f(x)}{(x-a)g(x)} = \lim_{x\to a}\frac{f(x)}{g(x)}$ Note: Canceling out the common factor $(x-a)$ is a correct operation because $\lim_{x\to a}$ means that x approaches a but is not equal to a.	Ex 3. Compute $\lim_{x\to 1}\frac{x^2+x-2}{x-1}$. Let's try substitution: $\lim_{x\to 1}\frac{x^2+x-2}{x-1} = \frac{1^2+1-2}{1-1} = \frac{0}{0}$ (indeterminate case) First, we have to cancel out the common factor: $\lim_{x\to 1}\frac{x^2+x-2}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1}(x+2)$ Second, we can use substitution: $\lim_{x\to 1}(x+2) = 1+2 = 3$ $\therefore \lim_{x\to 1}\frac{x^2+x-2}{x-1} = 3$
D Conjugate Radicals When dealing with the indeterminate form $\frac{0}{0}$ you may use the *conjugate radicals* to cancel out the common factor that generates zeros.	Ex 4. Compute $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$. Substitution does not work (case $\frac{0}{0}$). $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x\to 0}\frac{(x+4)-2^2}{x(\sqrt{x+4}+2)} =$ $= \lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)} = \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$

E Change of Variables
By changing the variable, the process of canceling the common factor may be simplified.

Note. If the change of variable is $u = g(x)$ then:
as $x \to a$, $u \to g(a)$

Ex 5. Change the variable to compute $\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

$\sqrt{x} = x^{\frac{1}{2}}, \quad \sqrt[3]{x} = x^{\frac{1}{3}}, \quad LCD = 6, \quad u = x^{\frac{1}{6}}$

$u^2 = (x^{\frac{1}{6}})^2 = x^{\frac{1}{3}} = \sqrt[3]{x}, \quad u^3 = (x^{\frac{1}{6}})^3 = x^{\frac{1}{2}} = \sqrt{x}$

$x \to 1 \quad \Rightarrow \quad u \to 1^{\frac{1}{6}} = 1$

$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \lim_{u\to 1}\frac{u^3-1}{u^2-1} = \lim_{u\to 1}\frac{(u-1)(u^2+u+1)}{(u-1)(u+1)} =$

$= \lim_{u\to 1}\frac{u^2+u+1}{u+1} = \frac{1^2+1+1}{1+1} = \frac{3}{2}, \quad \therefore \lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \frac{3}{2}$

F Absolute Value Function
When dealing with an absolute value function, rewrite it as piece-wise defined function according to:

$|f(x)| = \begin{cases} f(x), & f(x) \ge 0 \\ -f(x), & f(x) < 0 \end{cases}$

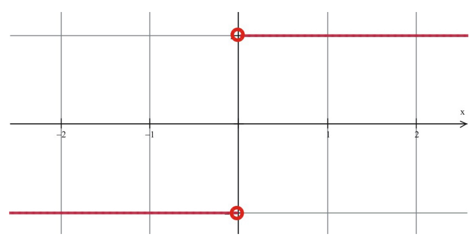
Ex 6. Compute each limit. Draw a diagram to illustrate.

a) $\lim_{x\to 0}\frac{|x|}{x}$

$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases} \Rightarrow \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

$\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}(-1) = -1, \quad \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1$

$\therefore \lim_{x\to 0}\frac{|x|}{x} = DNE$

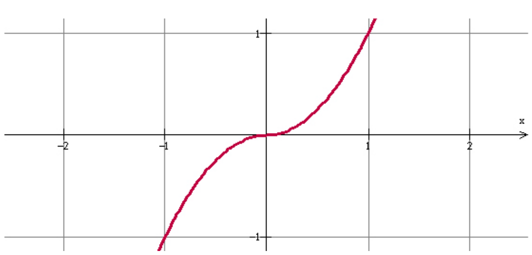


b) $\lim_{x\to 0} x|x|$

$x|x| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

$\lim_{x\to 0^-} x|x| = \lim_{x\to 0^-}(-x^2) = 0, \quad \lim_{x\to 0^+} x|x| = \lim_{x\to 0^+} x^2 = 0$

$\therefore \lim_{x\to 0} x|x| = 0$

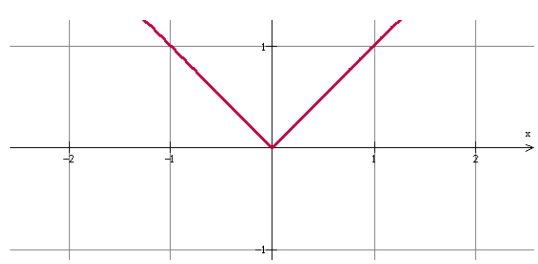


c) $\lim_{x\to 0}\frac{x^2}{|x|}$

$\frac{x^2}{|x|} = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$\lim_{x\to 0^-}\frac{x^2}{|x|} = \lim_{x\to 0^-} x = 0, \quad \lim_{x\to 0^+}\frac{x^2}{|x|} = \lim_{x\to 0^+}(-x) = 0$

$\therefore \lim_{x\to 0}\frac{x^2}{|x|} = 0$



Reading: Nelson Textbook, Pages 40-45
Homework: Nelson Textbook: Page 45 #4f, 7, 8, 9, 10, 13b, 14b, 16, 17