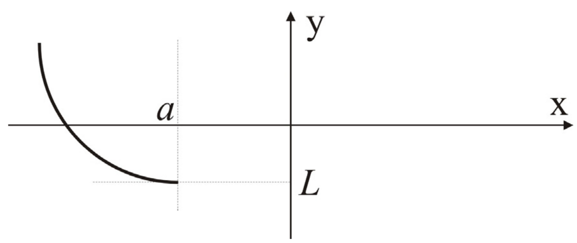
1.4 Limit of a Function

A Left-Hand Limit

If the values of $y = f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x < a$, then:

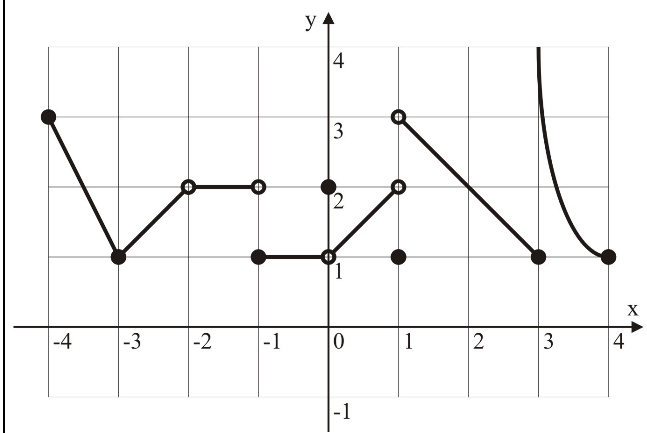
$$\lim_{x \to a^-} f(x) = L$$

Read: The limit of the function $f(x)$ as x approaches a from the left is L.



Notes:
1. The function *may be or not defined* at a.
2. DNE stands for *Does Not Exist*.
3. L must be a number.
4. ∞ is not a number.

Ex 1. Use the function $y = f(x)$ defined by the following graph to find each limit.



a) $\lim_{x \to -4^-} f(x) = DNE$

b) $\lim_{x \to -2^-} f(x) = 2$

c) $\lim_{x \to -1^-} f(x) = 2$

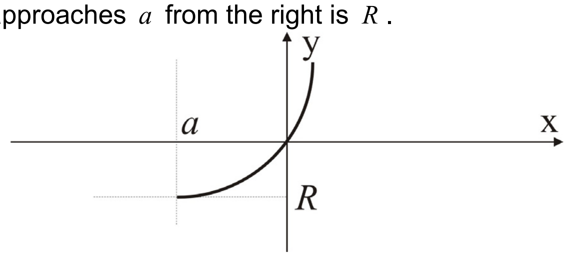
d) $\lim_{x \to 3^-} f(x) = 1$

B Right-Hand Limit

If the values of $y = f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x > a$, then:

$$\lim_{x \to a^+} f(x) = R$$

Read: The limit of the function $f(x)$ as x approaches a from the right is R.



Notes:
1. R must be a number. ∞ is not a number.
2. The function *may be or not defined* at a.

Ex 2. Use the function $y = f(x)$ defined at Ex 1. to find each limit.

a) $\lim_{x \to -1^+} f(x) = 1$

b) $\lim_{x \to 3^+} f(x) = \infty \quad (DNE)$

c) $\lim_{x \to 1^+} f(x) = 3$

d) $\lim_{x \to -2^+} f(x) = 2$

C Limit

If the values of $y = f(x)$ can be made arbitrarily close to l by taking x sufficiently close to a (*from both* sides), then:

$$\lim_{x \to a} f(x) = l$$

Read: The limit of the function $f(x)$ as x approaches a is l.

Ex 3. Use the function $y = f(x)$ defined at Ex 1. to find each limit.

a) $\lim_{x \to -4} f(x) = DNE$ because $\lim_{x \to -4^-} f(x) = DNE$

b) $\lim_{x \to -1} f(x) = DNE$ because $\lim_{x \to -1^-} f(x) = 2 \neq \lim_{x \to -1^+} f(x) = 1$

c) $\lim_{x \to 3} f(x) = DNE$ because $\lim_{x \to 3^+} f(x) = \infty$

d) $\lim_{x \to -3} f(x) = 1$

e) $\lim_{x \to -2} f(x) = 2$

Notes:

1. If $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$ then $\lim\limits_{x \to a} f(x)$ does exist and $L = R = l$.
2. If $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$ then $\lim\limits_{x \to a} f(x)$ Does Not Exist (DNE).
3. l must be a number. ∞ is not a number.
4. The function *may be or not defined* at a.

f) $\lim\limits_{x \to 0} f(x) = 1$

D Substitution

If the function is defined by a *formula* (algebraic expression) then the limit of the function at a point a may be determined by *substitution*:

$$\lim_{x \to a} f(x) = f(a)$$

Notes:

1. In order to use substitution, the function must be defined *on both sides* of the number a.
2. Substitution does not work if you get one of the following 7 *indeterminate cases*:

$$\infty - \infty \quad 0 \times \infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^\infty \quad \infty^0 \quad 0^0$$

Ex 4. Compute each limit.

a) $\lim\limits_{x \to 1^-} \frac{x^2}{x+1} = \frac{1^2}{1+1} = \frac{1}{2}$

b) $\lim\limits_{x \to 1^+} \frac{x^2}{x+1} = \frac{1^2}{1+1} = \frac{1}{2}$

c) $\lim\limits_{x \to 1} \frac{x^2}{x+1} = \frac{1^2}{1+1} = \frac{1}{2}$

d) $\lim\limits_{x \to 2^-} \sqrt{x-2} = DNE$

e) $\lim\limits_{x \to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0$

f) $\lim\limits_{x \to 2} \sqrt{x-2} = DNE$

E Piece-wise defined functions

If the function changes formula at a then:

1. Use the appropriate formula to find first the *left-side* and the *right-side* limits.
2. Compare the left-side and the right-side limits to conclude about the limit of the function at a.

Example:

$$f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x > a \end{cases}$$

$L = f_1(a), \quad R = f_2(a)$ (if exist)

Ex 5. Consider $f(x) = \begin{cases} 2x-3, & x < 2 \\ 0, & x = 2 \\ x^2 - 1, & x > 2 \end{cases}$

a) Find $\lim\limits_{x \to 2} f(x)$.

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x-3) = (2)(2) - 3 = 1$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^2 - 1) = 2^2 - 1 = 3$

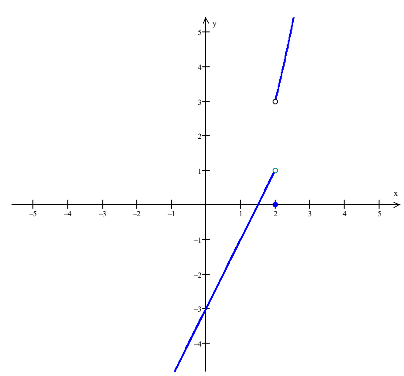
$\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x) \quad \Rightarrow \quad \therefore \lim\limits_{x \to 2} f(x) = DNE$

b) Find $\lim\limits_{x \to 0} f(x)$.

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (2x-3) = 2(0) - 3 = -3$

$\therefore \lim\limits_{x \to 0} f(x) = -3$

c) Draw a diagram to illustrate the situation.



Reading: Nelson Textbook, Pages 34-37

Homework: Nelson Textbook: Page 37 #4d, 5, 6, 7, 10cef, 11c, 15