1.4 Limit of a Function

**A Left-Hand Limit**
If the values of \( y = f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) with \( x < a \), then:

\[
\lim_{x \to a^-} f(x) = L
\]

Read: The limit of the function \( f(x) \) as \( x \) approaches \( a \) from the left is \( L \).

Notes:
1. The function may be or not defined at \( a \).
2. DNE stands for Does Not Exist.
3. \( L \) must be a number.
4. \( \infty \) is not a number.

**B Right-Hand Limit**
If the values of \( y = f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) with \( x > a \), then:

\[
\lim_{x \to a^+} f(x) = R
\]

Read: The limit of the function \( f(x) \) as \( x \) approaches \( a \) from the right is \( R \).

Notes:
1. \( R \) must be a number. \( \infty \) is not a number.
2. The function may be or not defined at \( a \).

**C Limit**
If the values of \( y = f(x) \) can be made arbitrarily close to \( l \) by taking \( x \) sufficiently close to \( a \) (from both sides), then:

\[
\lim_{x \to a} f(x) = l
\]

Read: The limit of the function \( f(x) \) as \( x \) approaches \( a \) is \( l \).

Ex 1. Use the function \( y = f(x) \) defined by the following graph to find each limit.

![Graph](image)

- a) \( \lim_{x \to 4^-} f(x) = DNE \)
- b) \( \lim_{x \to 2^-} f(x) = 2 \)
- c) \( \lim_{x \to 1^-} f(x) = 2 \)
- d) \( \lim_{x \to 3^-} f(x) = 1 \)

Ex 2. Use the function \( y = f(x) \) defined at Ex 1. to find each limit.

- a) \( \lim_{x \to -1} f(x) = 1 \)
- b) \( \lim_{x \to 3} f(x) = \infty \) (DNE)
- c) \( \lim_{x \to 1^+} f(x) = 3 \)
- d) \( \lim_{x \to 2^+} f(x) = 2 \)

Ex 3. Use the function \( y = f(x) \) defined at Ex 1. to find each limit.

- a) \( \lim_{x \to 4^-} f(x) = DNE \) because \( \lim_{x \to 4^-} f(x) = DNE \)
- b) \( \lim_{x \to 1^-} f(x) = 2 \neq \lim_{x \to 1^+} f(x) = 1 \)
- c) \( \lim_{x \to 3^-} f(x) = DNE \) because \( \lim_{x \to 3^-} f(x) = \infty \)
- d) \( \lim_{x \to 3^+} f(x) = 1 \)
- e) \( \lim_{x \to 2^+} f(x) = 2 \)
Notes:
1. If \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \) then \( \lim_{x \to a} f(x) \) does exist and \( L = R = L \).
2. If \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \) then \( \lim_{x \to a} f(x) \) Does Not Exist (DNE).
3. \( L \) must be a number. \( \infty \) is not a number.
4. The function may be or not defined at \( a \).

**D Substitution**
If the function is defined by a formula (algebraic expression) then the limit of the function at a point \( a \) may be determined by substitution:
\[
\lim_{x \to a} f(x) = f(a)
\]
Notes:
1. In order to use substitution, the function must be defined on both sides of the number \( a \).
2. Substitution does not work if you get one of the following 7 indeterminate cases:
   \( \frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 0^0, \infty^0, 0^0, \infty^\infty \)

**E Piece-wise defined functions**
If the function changes formula at \( a \) then:
1. Use the appropriate formula to find first the left-side and the right-side limits.
2. Compare the left-side and the right-side limits to conclude about the limit of the function at \( a \).
Example:
\[
f(x) = \begin{cases} 
   f_1(x), & x < a \\
   f_2(x), & x > a 
\end{cases}
\]
\[
L = f_1(a), \quad R = f_2(a) \quad \text{ (if exist)}
\]

Ex 4. Compute each limit.

a) \( \lim_{x \to 1} \frac{x^2}{x+1} = \frac{1}{2} \)

b) \( \lim_{x \to 1} x^2 = 1 \)

c) \( \lim_{x \to 1} \frac{x^2}{x+1} = \frac{1}{2} \)

d) \( \lim_{x \to 2} \sqrt{x-2} = DNE \)

e) \( \lim_{x \to 2} \sqrt{x-2} = \sqrt{2-2} = 0 \)

Ex 5. Consider \( f(x) = \begin{cases} 
   2x-3, & x < 2 \\
   0, & x = 2 \\
   x^2-1, & x > 2 
\end{cases} \)

a) Find \( \lim_{x \to 2} f(x) \).
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x-3) = 2(2)-3 = 1
\]
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2-1) = 2^2-1 = 3
\]
\[
\lim_{x \to 2} f(x) \neq \lim_{x \to 2} f(x) \quad \Rightarrow \quad \lim_{x \to 2} f(x) = DNE
\]

b) Find \( \lim_{x \to 0} f(x) \).
\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} (2x-3) = 2(0)-3 = -3
\]
\[
\lim_{x \to 0} f(x) = -3
\]

c) Draw a diagram to illustrate the situation.

Reading: Nelson Textbook, Pages 34-37
Homework: Nelson Textbook: Page 37 #4d, 5, 6, 7, 10c, 11c, 15