### 1.3 Rate of Change

#### A Average Rate of Change

\[ y = f(x), \quad y_1 = f(x_1), \quad y_2 = f(x_2) \]
\[ \Delta x = x_2 - x_1 \quad (\text{change in variable } x) \]
\[ \Delta y = y_2 - y_1 \quad (\text{change in variable } y) \]

The **Average Rate of Change** (ARC) in \( y \) variable over the interval \( [x_1, x_2] \) is given by:

\[ ARC = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]

**Note:** The **Average Rate of Change** is the same as the **slope of the secant line** passing through the points \((x_1, y_1)\) and \((x_2, y_2)\).

If \( x_1 = a \) and \( x_2 = a + h \) then:

\[ ARC = \frac{f(a + h) - f(a)}{h} \]

**Ex 1.** Consider \( y = f(x) = (x+1)^2 \). Find the rate of change in the \( y \) variable over the interval \([-1, 2] \).

\[ x_1 = -1, \quad y_1 = f(x_1) = f(-1) = (-1+1)^2 = 0 \]
\[ x_2 = 2, \quad y_2 = f(x_2) = f(2) = (2+1)^2 = 9 \]
\[ \Delta x = x_2 - x_1 = 2 - (-1) = 3, \quad \Delta y = y_2 - y_1 = 9 - 0 = 9 \]
\[ ARC = \frac{\Delta y}{\Delta x} = \frac{9}{3} = 3 \quad \therefore ARC = 3 \]

#### B Average Velocity

Let \( s = s(t) \) be the position function, where \( s \) is position in meters and \( t \) is the time in seconds.

\[ s = s(t), \quad s_1 = s(t_1), \quad s_2 = s(t_2) \]
\[ \Delta t = t_2 - t_1 \quad (\text{time duration}) \]
\[ \Delta s = s_2 - s_1 \quad (\text{displacement}) \]

The **Average Velocity** (AV) over the time interval \([t_1, t_2]\) is given by:

\[ AV = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} \]

**Note:** The unit of measurement for velocity is \( \text{m/s} \).

**Ex 2.** A rock is launched vertically upward. The height of the rock is given by \( s(t) = 100t - 10t^2 \). Find the average velocity over the time interval \([1, 2]\).

\[ t_1 = 1, \quad s_1 = s(t_1) = s(1) = 100(1) - 10(1)^2 = 90 \text{m} \]
\[ t_2 = 2, \quad s_2 = s(t_2) = s(2) = 100(2) - 10(2)^2 = 160 \text{m} \]
\[ \Delta t = t_2 - t_1 = 2 - 1 = 1 \text{ s}, \quad \Delta s = s_2 - s_1 = 160 - 90 = 70 \text{ m} \]
\[ AV = \frac{\Delta s}{\Delta t} = \frac{70}{1} = 70 \quad \therefore AV = 70 \text{ m/s} \]

#### C Instantaneous Rate of Change

As \( h \to 0 \) the Average Rate of Change approaches to the **Instantaneous Rate of Change** (IRC):

\[ IRC = RC = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

**Note:** The **Instantaneous Rate of Change** (IRC) is the same as the **slope of the tangent line** at the point \((a, f(a))\).

Similarly, the **Average Velocity** (AV) approaches **Instantaneous Velocity** (IV):

\[ IV = v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \]

**Ex 3.** Consider the following position function: \( s(t) = t^2 - 4t \).

a) Find the instantaneous velocity at \( t = 3 \text{ s} \).

\[ a = 3, \quad s(a) = s(3) = 3^2 - 4(3) = -3 \]
\[ s(a+h) = s(3+h) = (3+h)^2 - 4(3+h) = 9 + 6h + h^2 - 12 - 4h \]
\[ v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \to 0} \frac{(9 + 6h + h^2 - 12 - 4h) - (-3)}{h} = \]
\[ = \lim_{h \to 0} \frac{h(2+h)}{h} = \lim_{h \to 0} (2+h) = 2 + 0 = 2 \quad \therefore v = 2 \text{ m/s} \]

b) Find the instantaneous velocity at the generic moment \( t = a \)

\[ s(a) = a^2 - 4a \]
\[ s(a+h) = (a+h)^2 - 4(a+h) = a^2 + 2ah + h^2 - 4a - 4h \]
\[ v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \to 0} \frac{(a^2 + 2ah + h^2 - 4a - 4h) - (a^2 - 4a)}{h} = \]
\[ = \lim_{h \to 0} \frac{h(2a + h - 4)}{h} = \lim_{h \to 0} (2a + h - 4) = 2a + 0 - 4 = 2a - 4 \]
\[ \therefore v = 2a - 4 \]
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**c)** Use the formula at part b) to compute the velocity at time \( t = 5s \).

\[
\begin{align*}
  v &= 2a - 4 \\
  a &= 5s \\
  \Rightarrow v &= 2(5) - 4 \\
  \therefore v &= 6m/s
\end{align*}
\]

**d)** Find the moment(s) of time at which the velocity is zero.

\[
\begin{align*}
  v &= 2a - 4 \\
  v &= 0 \\
  \Rightarrow 0 &= 2a - 4 \\
  \therefore a &= 2s
\end{align*}
\]

**Ex 4.** A spherical balloon is inflated. Find the instantaneous rate of change in volume of the balloon with respect to its radius when the radius is 10 m.

\[
V(r) = \frac{4\pi}{3} r^3, \quad r = a = 10m
\]

\[
V(a) = V(10) = \frac{4\pi}{3} (10)^3, \quad V(a + h) = V(10 + h) = \frac{4\pi}{3} (10 + h)^3
\]

\[
IRC = \lim_{h \to 0} \frac{\Delta V}{\Delta r} = \lim_{h \to 0} \frac{V(a + h) - V(a)}{h} = \lim_{h \to 0} \frac{\frac{4\pi}{3} (10 + h)^3 - \frac{4\pi}{3} (10)^3}{h}
\]

\[
= \frac{4\pi}{3} \lim_{h \to 0} \left( \frac{(10 + h)^3 - (10)^3}{h} \right) = \frac{4\pi}{3} \lim_{h \to 0} \frac{h[(10 + h)^2 + (10 + h)(10) + 10^2]}{h}
\]

\[
= \frac{4\pi}{3} \lim_{h \to 0} [(10 + h)^2 + (10 + h)(10) + 10^2] = \frac{4\pi}{3} [(10 + 0)^2 + (10 + 0)(10) + 10^2]
\]

\[
\therefore IRC = 400\pi \ m^2
\]

**Reading:** Nelson Textbook, Pages 22-28

**Homework:** Nelson Textbook: Page 28 #2a, 7, 12, 14, 15b, 20, 22