1.3 Rate of Change

A Average Rate of Change $y = f(x), \quad y_1 = f(x_1), \quad y_2 = f(x_2)$ $\Delta x = x_2 - x_1$ (change in variable x) $\Delta y = y_2 - y_1$ (change in variable y) The *Average Rate of Change* (ARC) in y variable over the interval $[x_1, x_2]$ is given by: $ARC = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ Note: The *Average Rate of Change* is the same as the *slope of the secant line* passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. If $x_1 = a$ and $x_2 = a + h$ then: $ARC = \frac{f(a+h) - f(a)}{h}$	Ex 1. Consider $y = f(x) = (x+1)^2$. Find the rate of change in the y variable over the interval $[-1,2]$. $x_1 = -1, \quad y_1 = f(x_1) = f(-1) = (-1+1)^2 = 0$ $x_2 = 2, \quad y_2 = f(x_2) = f(2) = (2+1)^2 = 9$ $\Delta x = x_2 - x_1 = 2 - (-1) = 3, \quad \Delta y = y_2 - y_1 = 9 - 0 = 9$ $ARC = \frac{\Delta y}{\Delta x} = \frac{9}{3} = 3 \quad \therefore ARC = 3$
B Average Velocity Let $s = s(t)$ be the position function, where s is position in meters and t is the time in seconds. $s = s(t), \quad s_1 = s(t_1), \quad s_2 = s(t_2)$ $\Delta t = t_2 - t_1$ (time duration) $\Delta s = s_2 - s_1$ (displacement) The *Average Velocity* (AV) over the time interval $[t_1, t_2]$ is given by: $AV = \frac{rise}{run} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ Note: The *unit* of measurement for velocity is m/s.	Ex 2. A rock is launched vertically upward. The height of the rock is given by $s(t) = 100t - 10t^2$. Find the average velocity over the time interval $[1,2]$. $t_1 = 1, \quad s_1 = s(t_1) = s(1) = 100(1) - 10(1)^2 = 90m$ $t_2 = 2, \quad s_2 = s(t_2) = s(2) = 100(2) - 10(2)^2 = 160m$ $\Delta t = t_2 - t_1 = 2 - 1 = 1s, \quad \Delta s = s_2 - s_1 = 160 - 90 = 70m$ $AV = \frac{\Delta s}{\Delta t} = \frac{70}{1} = 70 \quad \therefore AV = 70m/s$
C Instantaneous Rate of Change As $h \to 0$ the Average Rate of Change approaches to the *Instantaneous Rate of Change* (IRC): $IRC = RC = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ Note: The *Instantaneous Rate of Change* (IRC) is the same as the *slope of the tangent line* at the point $P(a, f(a))$. Similarly, the *Average Velocity* (AV) approaches *Instantaneous Velocity* (IV): $IV = v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$	Ex 3. Consider the following position function: $s(t) = t^2 - 4t$. a) Find the instantaneous velocity at $t = 3s$. $a = 3, \quad s(a) = s(3) = 3^2 - 4(3) = -3$ $s(a+h) = s(3+h) = (3+h)^2 - 4(3+h) = 9 + 6h + h^2 - 12 - 4h$ $v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \to 0} \frac{(9 + 6h + h^2 - 12 - 4h) - (-3)}{h} =$ $= \lim_{h \to 0} \frac{h(2+h)}{h} = \lim_{h \to 0}(2+h) = 2 + 0 = 2 \quad \therefore v = 2m/s$ b) Find the instantaneous velocity at the generic moment $t = a$ $s(a) = a^2 - 4a$ $s(a+h) = (a+h)^2 - 4(a+h) = a^2 + 2ah + h^2 - 4a - 4h$ $v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \to 0} \frac{(a^2 + 2ah + h^2 - 4a - 4h) - (a^2 - 4a)}{h} =$ $= \lim_{h \to 0} \frac{h(2a + h - 4)}{h} = \lim_{h \to 0}(2a + h - 4) = 2a + 0 - 4 = 2a - 4$ $\therefore v = 2a - 4$

c) Use the formula at part b) to compute the velocity at time $t = 5s$.

$$\begin{cases} v = 2a - 4 \\ a = 5s \end{cases} \Rightarrow v = 2(5) - 4 = 6 \quad \therefore v = 6m/s$$

d) Find the moment(s) of time at which the velocity is zero.

$$\begin{cases} v = 2a - 4 \\ v = 0 \end{cases} \Rightarrow 0 = 2a - 4 \Rightarrow a = 2 \quad \therefore a = 2s$$

Ex 4. A spherical balloon is inflated. Find the instantaneous rate of change in volume of the balloon with respect to its radius when the radius is $10m$.

$$V(r) = \frac{4\pi}{3} r^3, \quad r = a = 10m$$

$$V(a) = V(10) = \frac{4\pi}{3}(10)^3, \quad V(a+h) = V(10+h) = \frac{4\pi}{3}(10+h)^3$$

$$IRC = \lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = \lim_{h \to 0} \frac{V(a+h) - V(a)}{h} = \lim_{h \to 0} \frac{\frac{4\pi}{3}(10+h)^3 - \frac{4\pi}{3}(10)^3}{h} =$$

$$= \frac{4\pi}{3} \lim_{h \to 0} \frac{(10+h)^3 - (10)^3}{h} = \frac{4\pi}{3} \lim_{h \to 0} \frac{h[(10+h)^2 + (10+h)(10) + 10^2]}{h} =$$

$$= \frac{4\pi}{3} \lim_{h \to 0} [(10+h)^2 + (10+h)(10) + 10^2] = \frac{4\pi}{3}[(10+0)^2 + (10+0)(10) + 10^2]$$

$$\therefore IRC = 400\pi \ m^2$$

Reading: Nelson Textbook, Pages 22-28
Homework: Nelson Textbook: Page 28 #2a, 7, 12, 14, 15b, 20, 22

