1.2 The Slope of the Tangent

A Lines

The slope of a line: \( m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \)

Slope y-intercept equation of a line: \( y = mx + b \)

Slope-point equation of a line: \( y - y_1 = m(x - x_1) \)

For parallel lines, slopes are equal: \( m_1 = m_2 \)

For perpendicular lines, slopes are negative reciprocal: \( m_2 = -\frac{1}{m_1} \) or \( m_1m_2 = -1 \)

Ex 1. The equation of the line \( L_1 \) is: \( 2x - 3y + 6 = 0 \).

a) Find the slope of the line \( L_1 \).
\[ y = \frac{2}{3}x + 2 \Rightarrow m_1 = \frac{2}{3} \]

b) Find the equation of the line \( L_2 \), that is parallel to the line \( L_1 \) and passes through the point \( P(2,-1) \).
\[ y - (-1) = \frac{2}{3}(x - 2) \Rightarrow y = \frac{2}{3}x - 1 - \frac{4}{3} \Rightarrow \]
\[ \therefore L_2 : y = \frac{2}{3}x - \frac{7}{3} \]

c) Find the equation of the line \( L_3 \), that is perpendicular to the line \( L_1 \) and passes through the point \( Q(4,2) \).
\[ m_3 = -\frac{1}{m_1} = -\frac{3}{2} \Rightarrow y - 2 = -\frac{3}{2}(x - 4) \Rightarrow \]
\[ \therefore L_3 : y = -\frac{3}{2}x + 8 \]

B Function Notation

\[ y = f(x) \]

where:
- \( x \) is the argument, input or independent variable
- \( y \) is the value, output or dependant variable
- \( f \) is the name of the function.

Ex 2. Let \( f(x) = \sqrt{x^2 - 1} \). Find:

a) \( f(1) \)
\[ f(1) = \sqrt{1^2 - 1} = 0 \Rightarrow \therefore f(1) = 0 \]

b) \( f(a) \)
\[ \therefore f(a) = \sqrt{a^2 - 1} \]

c) \( f(a + 1) \)
\[ f(a + 1) = \sqrt{(a + 1)^2 - 1} = \sqrt{a^2 + 2a + 1 - 1} = \sqrt{a^2 + 2a} \]
\[ \therefore f(a + 1) = \sqrt{a^2 + 2a} \]

C Secant Line

Let \( y = f(x) \) be a function and \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) two points on its graph.

The slope of the secant line that passes through the points \( P \) and \( Q \) is given by:
\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]

Ex 3. Consider \( f(x) = \frac{2}{x + 1} \). Find the equation of the secant line that passes through the points \( A(0,2) \) and \( B(-3,-1) \).
\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - 2}{(-3) - (0)} = 1 \]
\[ y - 2 = 1(x - 0) \]
\[ \therefore y = x + 2 \]
If \( P(a, f(a)) \) and \( Q(a + h, f(a + h)) \) then the slope of the secant line is given by:

\[
m = \frac{f(a + h) - f(a)}{h}
\]

### D Tangent Line

As the point \( Q \) approaches the point \( P \), the secant line approaches the tangent line at \( P \). See the diagram on the right side.

The slope of the tangent line at \( P(a, f(a)) \) is:

\[
m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{1}
\]

### E Graphical Computation

1. Draw the tangent line using a ruler
2. Choose two points on the tangent line \( A(x_1, y_1) \) and \( B(x_2, y_2) \)
3. Use the formula:

\[
m \approx \frac{y_2 - y_1}{x_2 - x_1}
\]

To estimate the slope of the tangent line.

Note. The slope of the tangent line can be calculated using:

\[
m = \tan \alpha
\]

where \( \alpha \) is the angle between the tangent line and the positive direction of the x-axis.

### F Numerical Computation

1. Use the formula \( m \approx \frac{f(a + h) - f(a)}{h} \) \( \tag{2} \)
2. Choose a sequence \( h_1, h_2, h_3, \ldots \to 0 \).
3. Compute \( m_1, m_2, m_3, \ldots \)
4. Observe the pattern and conclude.
5. Be careful at "difference catastrophe".

Note. The difference catastrophe is related to the limited capacity of memorizing numbers by any technologic device (scientific calculator, computer, etc). If two numbers are very close, then they have the same internal representation in the memory.

Ex 4. Find the slope of the tangent line to the curve \( y = 2\sqrt{x} - 1 \) at the point \( P(5,4) \).

Ex 5. Consider \( y = f(x) = x^3 \). Estimate numerically the slope of the tangent line at \( P(1,1) \) using

\[
h_1 = 0.1, h_2 = 0.0001, h_3 = 10^{-20}.
\]

\( a = 1, f(1) = 1 \)

\[
m_1 \approx \frac{(1+0.1)^3 - 1}{0.1} \approx 3.31
\]

\[
m_2 \approx \frac{(1+0.0001)^3 - 1}{0.0001} \approx 3.0003
\]

\[
m_3 \approx \frac{(1+10^{-20})^3 - 1}{10^{-20}} \approx 0 \text{ (difference catastrophe)}
\]

\[\therefore m \approx 3\]
G Algebraic Computation
1. Use the formula \( m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \).
2. Do not substitute \( h \) by 0 because you will get the indeterminate case \( \frac{0}{0} \).
3. Compute algebraic the difference quotient \( DQ = \frac{f(a+h) - f(a)}{h} \) until you succeed to cancel out the factor \( h \).
4. Substitute in the remaining expression \( h \) by 0.

Ex 6. Find the slope of the tangent line to the graph of \( y = f(x) = x^2 - 3x \) at the point \( P(1,-2) \).

\[ a = 1, \ f(1) = -2 \]
\[ f(a + h) = (1 + h)^2 - 3(1 + h) = 1 + 2h + h^2 - 3 - 3h = -2 - h + h^2 \]
\[ DQ = \frac{f(a+h) - f(a)}{h} = \frac{f(1+h) - f(1)}{h} \]
\[ = \frac{(-2 + h^2) - (-2)}{h} = \frac{h^2 - h}{h} = h - 1 \]
\[ m = \lim_{h \to 0} (h - 1) = 0 - 1 - 1 \Rightarrow \therefore m = -1 \]

Ex 7. Find the equation of the tangent line to the graph of \( y = f(x) = \sqrt[3]{x-1} \) at the point \( P(2,1) \).

\( a = 2, \ \ f(a) = f(2) = 1 \)
\[ f(a + h) = f(2 + h) = \frac{1}{\sqrt[3]{2+h-1}} = \frac{1}{1+h} \]
\[ DQ = \frac{f(a+h) - f(a)}{h} = \frac{1}{\sqrt[3]{1+h}} - \frac{1}{1+h} - \frac{1}{1+h} + 1 \]
\[ = \frac{1}{1+h} - \frac{h}{(1+h)(\sqrt[3]{1+h} + 1)} - \frac{1}{(1+h)(\sqrt[3]{1+h} + 1)} + 1 \]
\[ m = \lim_{x \to 0} \frac{1}{(1+h)(\sqrt[3]{1+h} + 1)} - \frac{1}{(1+h)(\sqrt[3]{1+h} + 1)} + 1 \]
\[ = \frac{1}{2} \]
\[ \{ y -1 = 1 \} \Rightarrow \therefore y = -\frac{1}{2}x + 2 \]
\( P(2,1) \)

Reading: Nelson Textbook, Pages 10-18
Homework: Nelson Textbook: Page 18, #1a, 2a, 3ac, 4a, 5a, 6a, 8a, 9a, 10a, 11a, 15, 21, 25