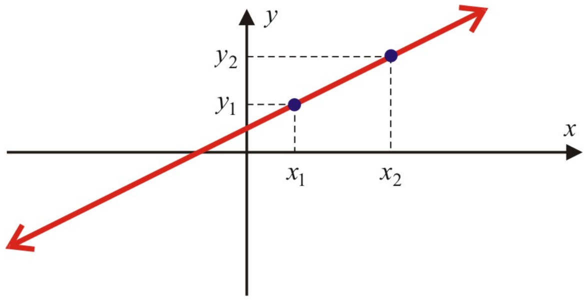
1.2 The Slope of the Tangent

A Lines



The *slope* of a line: $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Slope y-intercept equation of a line: $y = mx + b$

Slope-point equation of a line: $y - y_1 = m(x - x_1)$

For *parallel* lines, slopes are equal: $m_1 = m_2$

For *perpendicular* lines, slopes are negative reciprocal:

$$m_2 = -\frac{1}{m_1} \text{ or } m_1 m_2 = -1$$

Ex 1. The equation of the line L_1 is: $2x - 3y + 6 = 0$.

a) Find the slope of the line L_1.

$y = \frac{2}{3}x + 2 \quad \Rightarrow \quad \therefore m_1 = \frac{2}{3}$

b) Find the equation of the line L_2, that is parallel to the line L_1 and passes through the point $P(2,-1)$.

$y - (-1) = \frac{2}{3}(x - 2) \quad \Rightarrow \quad y = \frac{2}{3}x - 1 - \frac{4}{3} \quad \Rightarrow$

$\therefore L_2 : y = \frac{2}{3}x - \frac{7}{3}$

c) Find the equation of the line L_3, that is perpendicular to the line L_1 and passes through the point $Q(4,2)$.

$m_3 = -\frac{1}{m_1} = -\frac{3}{2} \quad \Rightarrow \quad y - 2 = -\frac{3}{2}(x - 4) \quad \Rightarrow$

$\therefore L_3 : y = -\frac{3}{2}x + 8$

B Function Notation

$$y = f(x)$$

where:
x is the *argument*, input or independent variable
y is the *value*, output or dependant variable
f is the *name* of the function.

Ex 2. Let $f(x) = \sqrt{x^2 - 1}$. Find:

a) $f(1)$

$f(1) = \sqrt{1^2 - 1} = 0 \quad \Rightarrow \quad \therefore f(1) = 0$

b) $f(a)$

$\therefore f(a) = \sqrt{a^2 - 1}$

c) $f(a+1)$

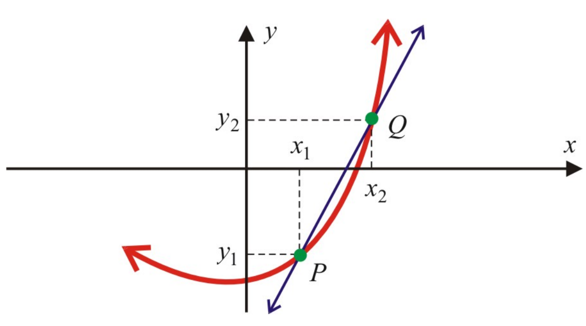
$f(a+1) = \sqrt{(a+1)^2 - 1} = \sqrt{a^2 + 2a + 1 - 1} = \sqrt{a^2 + 2a}$

$\therefore f(a+1) = \sqrt{a^2 + 2a}$

C Secant Line

Let $y = f(x)$ be a function and $P(x_1, y_1)$ and $Q(x_2, y_2)$ two points on its graph.
The *slope* of the *secant line* that passes through the points P and Q is given by:

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Ex 3. Consider $f(x) = \frac{2}{x+1}$. Find the equation of the secant line that passes through the points $A(0,2)$ and $B(-3,-1)$.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (2)}{(-3) - (0)} = 1$

$y - 2 = 1(x - 0)$

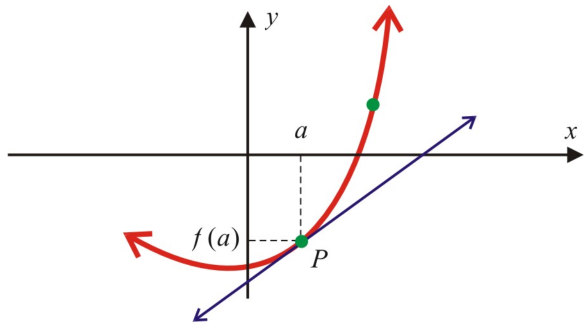
$\therefore y = x + 2$

If $P(a, f(a))$ and $Q(a+h, f(a+h))$ then the *slope* of the *secant line* is given by:

$$m = \frac{f(a+h) - f(a)}{h}$$

D Tangent Line
As the point Q approaches the point P, the secant line approaches the *tangent line* at P. See the diagram on the right side.
The *slope* of the *tangent line* at $P(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (1)$$



E Graphical Computation
1. Draw the tangent line using a ruler
2. Choose two points on the tangent line $A(x_1, y_1)$ and $B(x_2, y_2)$
3. Use the formula:

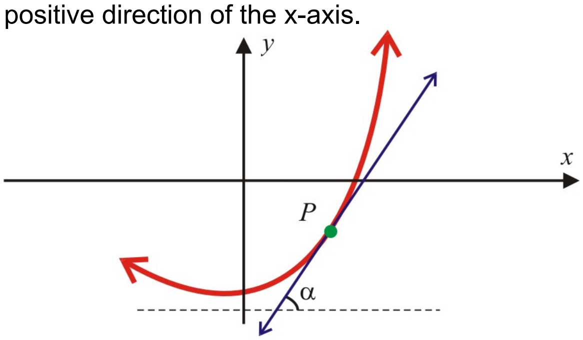
$$m \cong \frac{y_2 - y_1}{x_2 - x_1}$$

to estimate the slope of the tangent line.

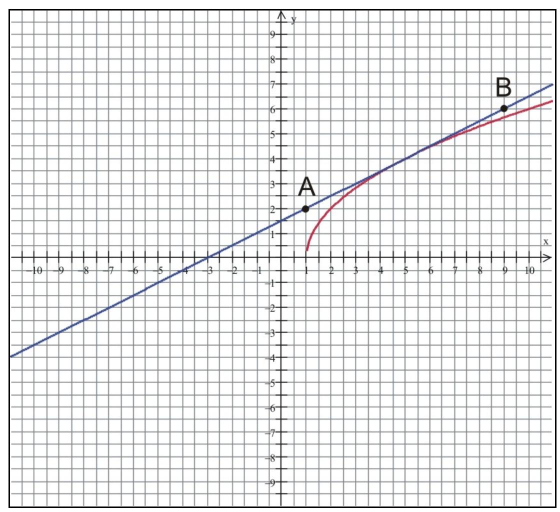
Note. The *slope* of the tangent line can be calculated using:

$$m = \tan \alpha$$

where α is the *angle* between the tangent line and the positive direction of the x-axis.



Ex 4. Find the slope of the tangent line to the curve $y = 2\sqrt{x-1}$ at the point $P(5,4)$.



$A(1,2), \quad B(9,6)$

$x_1 = 1, y_1 = 2, x_2 = 9, y_2 = 6$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{9-1} = \frac{1}{2} \quad \therefore m = \frac{1}{2}$$

F Numerical Computation
1. Use the formula $m \cong \dfrac{f(a+h) - f(a)}{h}$ (2)
2. Choose a sequence $h_1, h_2, h_3, \ldots \to 0$.
3. Compute $m_1, m_2, m_3, \ldots$
4. Observe the pattern and conclude.
5. Be careful at "difference catastrophe".

Note. The difference catastrophe is related to the limited capacity of memorizing numbers by any technologic device (scientific calculator, computer, etc). If two numbers are very close, then they have the same internal representation in the memory.

Ex 5. Consider $y = f(x) = x^3$. Estimate numerically the slope of the tangent line at $P(1,1)$ using

$h_1 = 0.1, h_2 = 0.0001, h_3 = 10^{-20}$.

$a = 1, f(1) = 1$

$$m_1 \cong \frac{(1+0.1)^3 - 1}{0.1} \cong 3.31$$

$$m_2 \cong \frac{(1+0.0001)^3 - 1}{0.0001} \cong 3.0003$$

$$m_3 \cong \frac{(1+10^{-20})^3 - 1}{10^{-20}} \cong 0 \; (\textit{difference catastrophe})$$

$\therefore m \cong 3$

G Algebraic Computation

1. Use the formula $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
2. Do not substitute h by 0 because you will get the indeterminate case $\frac{0}{0}$.
3. Compute algebraic the *difference quotient* $DQ = \frac{f(a+h) - f(a)}{h}$ until you succeed to cancel out the factor h.
4. *Substitute* in the remaining expression h by 0.

Ex 6. Find the slope of the tangent line to the graph of $y = f(x) = x^2 - 3x$ at the point $P(1,-2)$.

$a = 1, f(1) = -2$

$f(a+h) = f(1+h) = (1+h)^2 - 3(1+h) =$

$= 1 + 2h + h^2 - 3 - 3h = -2 - h + h^2$

$$DQ = \frac{f(a+h) - f(a)}{h} = \frac{f(1+h) - f(1)}{h} =$$

$$= \frac{(-2 - h + h^2) - (-2)}{h} = \frac{h^2 - h}{h} = h - 1$$

$$m = \lim_{h \to 0}(h-1) = 0 - 1 = -1 \quad \Rightarrow \quad \therefore m = -1$$

Ex 7. Find the equation of the tangent line to the graph of $y = f(x) = \frac{1}{\sqrt{x-1}}$ at the point $P(2,1)$.

$a = 2, \quad f(a) = f(2) = 1$

$$f(a+h) = f(2+h) = \frac{1}{\sqrt{2+h-1}} = \frac{1}{\sqrt{1+h}}$$

$$DQ = \frac{f(a+h) - f(a)}{h} = \frac{\frac{1}{\sqrt{1+h}} - 1}{h} = \frac{\frac{1}{\sqrt{1+h}} - 1}{h} \frac{\frac{1}{\sqrt{1+h}} + 1}{\frac{1}{\sqrt{1+h}} + 1}$$

$$= \frac{\frac{1}{1+h} - 1}{h\left(\frac{1}{\sqrt{1+h}} + 1\right)} = \frac{-h}{h(1+h)\left(\frac{1}{\sqrt{1+h}} + 1\right)} = \frac{-1}{(1+h)\left(\frac{1}{\sqrt{1+h}} + 1\right)}$$

$$m = \lim_{x \to 0} \frac{-1}{(1+h)\left(\frac{1}{\sqrt{1+h}} + 1\right)} = \frac{-1}{(1+0)\left(\frac{1}{\sqrt{1+0}} + 1\right)} = -\frac{1}{2}$$

$$\begin{cases} m = -\frac{1}{2} \\ P(2,1) \end{cases} \Rightarrow y - 1 = -\frac{1}{2}(x-2) \Rightarrow \therefore y = -\frac{1}{2}x + 2$$

Ex 8. Consider $y = f(x) = x^2 - 2x$.

a) Find the slope of the tangent line at the generic point $P(a, f(a))$.

$f(a) = a^2 - 2a$

$f(a+h) = (a+h)^2 - 2(a+h) = a^2 + 2ah + h^2 - 2a - 2h$

$$DQ = \frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h) - (a^2 - 2a)}{h}$$

$$= \frac{2ah + h^2 - 2h}{h} = 2a + h - 2$$

$$m = \lim_{x \to 0}(2a + h - 2) = 2a - 2 \quad \Rightarrow \quad \therefore m = 2a - 2$$

b) Find the point where the tangent line is horizontal.

$$\begin{cases} m = 0 \\ m = 2a - 2 \end{cases} \Rightarrow 0 = 2a - 2 \Rightarrow a = 1, f(1) = -1 \Rightarrow \quad \therefore P(1,-1)$$

c) Find the point P such that $m_P = 2$.

$$2a - 2 = 2 \quad \Rightarrow \quad a = 2, f(2) = 0 \quad \Rightarrow \quad \therefore P(2,0)$$

d) Find the point P such that the tangent line at P is perpendicular to the line $L_2 : x - 3y = 3$.

$$L_2 : y = \frac{1}{3}x - 1 \Rightarrow m_2 = \frac{1}{3} \Rightarrow m = -\frac{1}{m_2} = -3$$

$$\begin{cases} m = -3 \\ m = 2a - 2 \end{cases} \Rightarrow -3 = 2a - 2 \Rightarrow a = -\frac{1}{2}, f\left(-\frac{1}{2}\right) = \frac{1}{4} + 1 = \frac{5}{4}$$

$$\therefore P\left(-\frac{1}{2}, \frac{5}{4}\right)$$

Reading: Nelson Textbook, Pages 10-18
Homework: Nelson Textbook: Page 18, #1a, 2a, 3ac, 4a, 5a, 6a, 8a, 9a, 10a, 11a, 15, 21, 25